EEE321 Electromagnetic Fields and Waves

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(9^{th} Week)

Outline

- ***vxB*** field inside a nonconductor
- Motional Electromotange: Faraday Induction Law
- Faraday Induction Law for time Dependent **B**
- Electri Field Strength **E** expressed in terms of Potential *V* and **A**
- **E**, $-\nabla V$, $\boldsymbol{\partial A}/\partial t$, and $vx\boldsymbol{B}$ felds
- Six key equations
- Magnetic Energy

vxB Field inside a Nonconductor

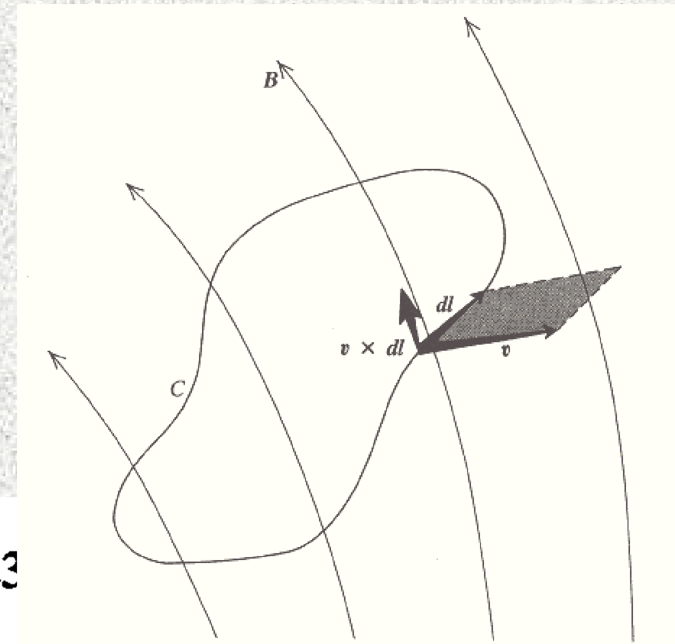
- (1) Suppose that nonmagnetic nonconductor moves in some arbitrary fashion in a constant magnetic field. Then a charge Q carried along inside the badyat a velocity ***v***, in a region where the magnetic flux is **B**, experiences a magnetic force $Q\boldsymbol{vxB}$.
- $\boldsymbol{vxB}$ has th edimension of **E**. The polarisation is therefore given by

$$\boldsymbol{P} = \epsilon_0 \chi_e (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \qquad (23\text{-}1)$$

- (2) if the noncoductor is magnetic, its equivalent currents follow the moving medium, but can be time-dependent if the ambient **B** is nonuniform. So the situation can be complex

Motional Electromotance. The Faraday Induction Law for vxB Field

- Consider a closed circuit C that moves as a whole and distorts in some arbitrary way in a constant magnetic field as in figüre



- Then the induced or motional electromotance is

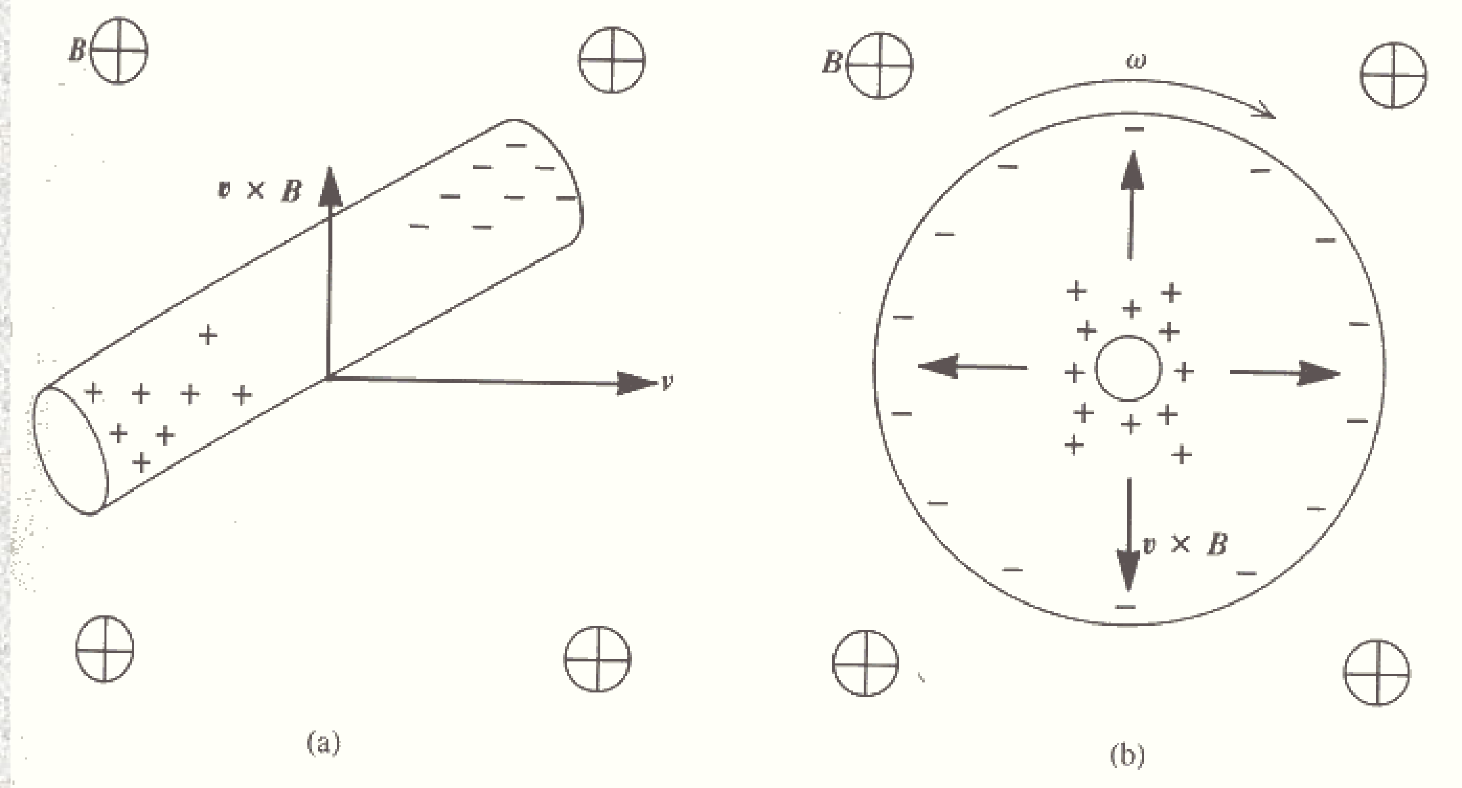
$$\mathcal{V} = \oint_C (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -\oint_C \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l}). \qquad (23$$

- $\boldsymbol{vxdl}$ is the area swept by the element ***dl*** in 1 second
- Integrating over the complate circuit, we find the induced eletromotance

$$\mathcal{V} = -\frac{d\Phi}{dt}. \qquad (23\text{-}3)$$

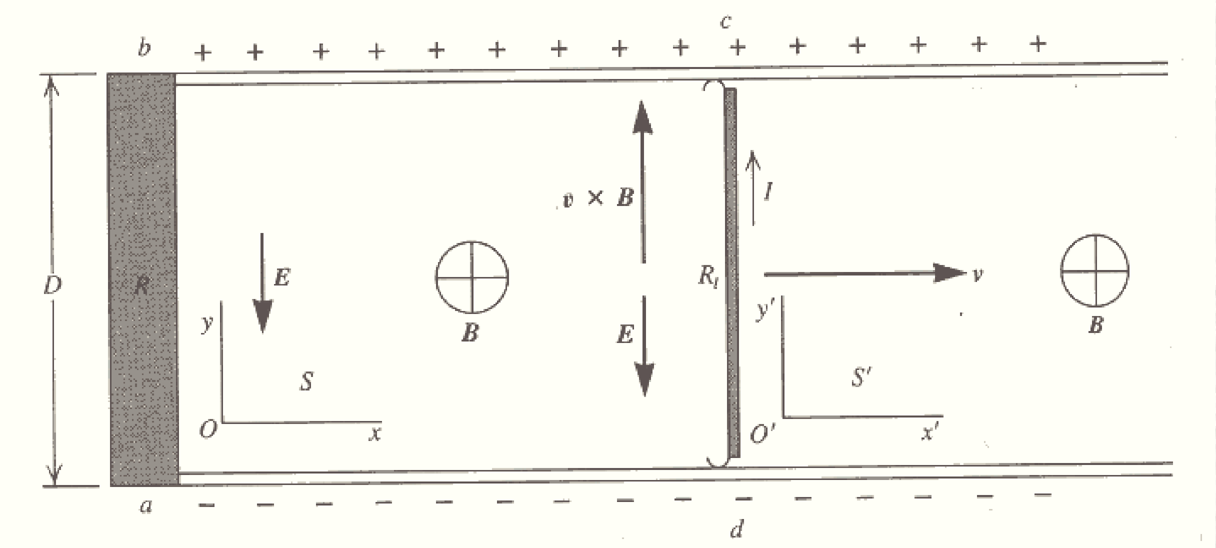
- This is the **Faraday induction law for $\boldsymbol{vxB}$** field.

- If C is open as in figüre, then the current flows until the electric field resulting from the accumulation of charge exactly cancels the $\boldsymbol{v x B}$ field.



Example: A simple-minded Generator

- An electric generator transforms mechanical energy to electric energy, usualy by moving conducting wires in a direction perpendicular to the magnetic filed.



(a)

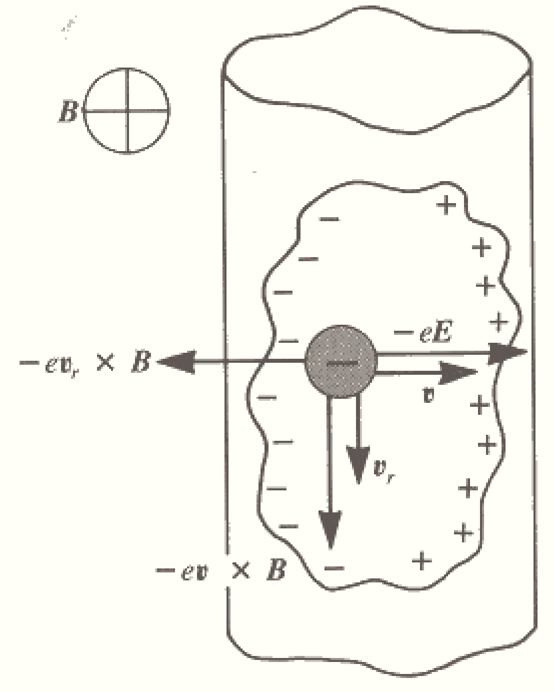
- The link slides to the right at a speed *v* such that $v^2 << c^2$ in a uniform **B**
- The resistance at the left-end of the line *R*, and that of the link is R_l. The gorizontal wires have zero resistance

- The eletromotance is

$$\mathcal{V} = -\frac{d\Phi}{dt} = BDv. \qquad (23\text{-}4)$$

- We have disregarded the magnetic flux resulting from the current I itself. In other Word R is large. Then

$$I = \frac{BDv}{R + R_l}. \qquad (23\text{-}5)$$



(b)

- In the fixed reference frame S, the force on a conducting electron of charge Q inside the link is $Q(\boldsymbol{E} + \boldsymbol{v}x\boldsymbol{B})$. Thus in the link,

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \sigma(-\nabla V + \boldsymbol{v} \times \boldsymbol{B}). \qquad (23\text{-}6)$$

- At tb *in figüre a,* $V_b = IR$. In either horizontal line $\boldsymbol{J} = \sigma \boldsymbol{E}$ is finite. Since $\sigma \rightarrow \infty$, by hypothesis, then $\mathbf{E} = 0$ and $\nabla V = 0$ and

$$V_d = V_a = 0, \qquad V_c = V_b = IR. \qquad (23\text{-}7)$$

- Inside *R* and R_l, with the y-axis an in figure

$$V = IR\frac{y}{D} = \frac{vBR}{R + R_l}y. \qquad (23\text{-}8)$$

- The voltage *Vc* across R_l is *IR*

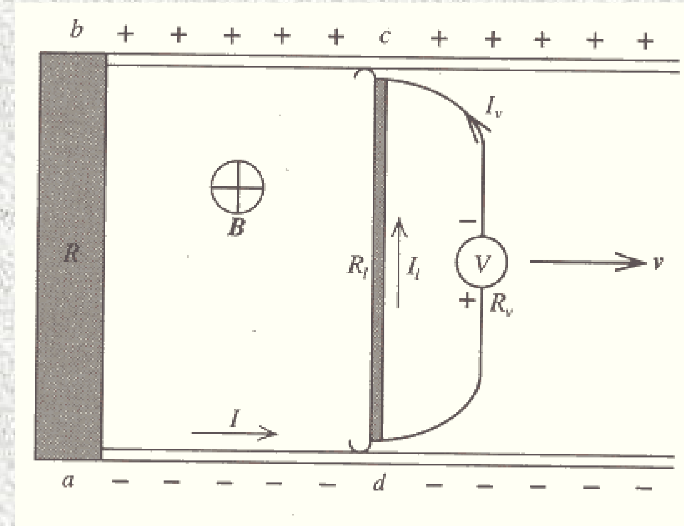
$$V_c = IR = I(R + R_l) - IR_l = vBD - IR_l. \qquad (23\text{-}9)$$

- This means that the motion generates a voltage *vBD* in the link, while its current causes a voltage drop IR_l.

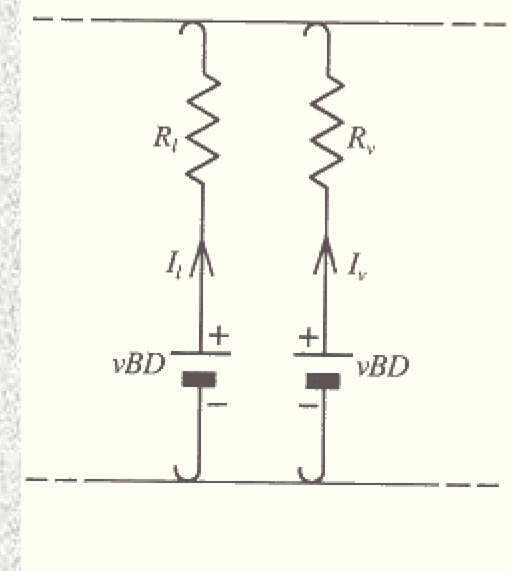
- Suppose we connect voltmeter across the link
- The resistance $R_v >> R_l$.
- If the current through the voltmeter is I_v, then the read voltage is $I_v R_v$.
- The magnetic field is $\boldsymbol{B} = -B\vec{\mathbf{z}}$. The current distribution that generates B is unspecified. Let us set

$$A_x = nBy, \qquad A_y = (n+1)Bx, \qquad (23\text{-}10)$$

- Where n is a pure number. If $n=0$ then the currents supplying **B** are all vertical.
- If $n=-1$, they are all horizontal



(a)



(b)

- Inside R and R_l

$$E = -\nabla V - \frac{\partial A}{\partial t} = -\nabla V \quad (23\text{-}11)$$

$$= -\frac{vBR}{R+R_l}\hat{y}. \quad (23\text{-}12)$$

- Let us now see what happens inside the link, in its own reference frame *S'*. Then

$$A'_x = A_x - \frac{vV}{c^2} = nBy - \frac{v^2}{c^2}\frac{BR}{R+R_l}y \approx nBy, \quad (23\text{-}13)$$

$$A'_y = A_y = (n+1)Bx = (n+1)Bvt, \quad (23\text{-}14)$$

$$V' = V - vA_x = \frac{vBR}{R+R_l}y - vnBy = \left(\frac{R}{R+R_l} - n\right)vBy. \quad (23\text{-}15)$$

$$V'_c = \left(\frac{R}{R+R_l} - n\right)vBD. \quad (23\text{-}16)$$

- Observe also the appearance of the $\partial A'/\partial t$ term in *S′*. Now

$$\boldsymbol{E}' = -\boldsymbol{\nabla}'V' - \frac{\partial \mathbf{A}'}{\partial t'}, \qquad (23\text{-}17)$$

- Where $$\boldsymbol{\nabla}' = \frac{\partial}{\partial y'}\hat{\mathbf{y}} = \frac{\partial}{\partial y}\hat{\mathbf{y}}', \qquad t' = t - \frac{v}{c^2}x = t - \frac{v^2}{c^2}t \approx t. \qquad (23\text{-}18)$$

- Thus

$$\boldsymbol{E}' = -\frac{\partial V'}{\partial y}\hat{\mathbf{y}} - \frac{\partial \mathbf{A}'}{\partial t} = -\frac{\partial V'}{\partial y}\hat{\mathbf{y}} - \frac{\partial A'_y}{\partial t}\hat{\mathbf{y}} \qquad (23\text{-}19)$$

$$= \left[-\left(\frac{R}{R+R_l} - n\right)vB - (n+1)vB\right]\hat{\mathbf{y}} \qquad (23\text{-}20)$$

$$= -\left(\frac{R}{R+R_l} - 1\right)vB\hat{\mathbf{y}} = \frac{R_l}{R+R_l}vB\hat{\mathbf{y}}. \qquad (23\text{-}21)$$

- This shows that, in the moving reference frame of the link ***E′*** is equal to ***E*** plus ***vxB***

Example: Alternating-Current Generator

- The loop of figure rotates at an angular velocity *w* in a uniform, constand **B.**
- We can calculate the induced eletromotance by two way.
- (1) ***vxB***

Along rigth-hand side of the loop

$$b\boldsymbol{v} \times \boldsymbol{B} = \frac{\omega a}{2} Bb \sin\theta\, \hat{\boldsymbol{x}} = \frac{\omega abB}{2} \sin\omega t\, \hat{\boldsymbol{x}}. \qquad (23\text{-}24)$$

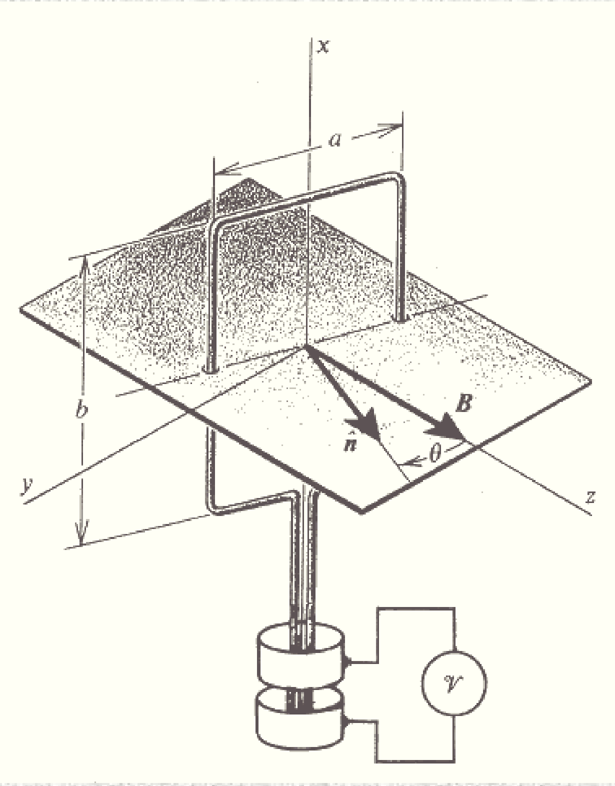
Along the left-hand side, it is same.

Along upper and lower side, ***vxB*** is perpendicular to the wire So

$$\mathcal{V} = abB\omega \sin\omega t. \qquad (23\text{-}25)$$

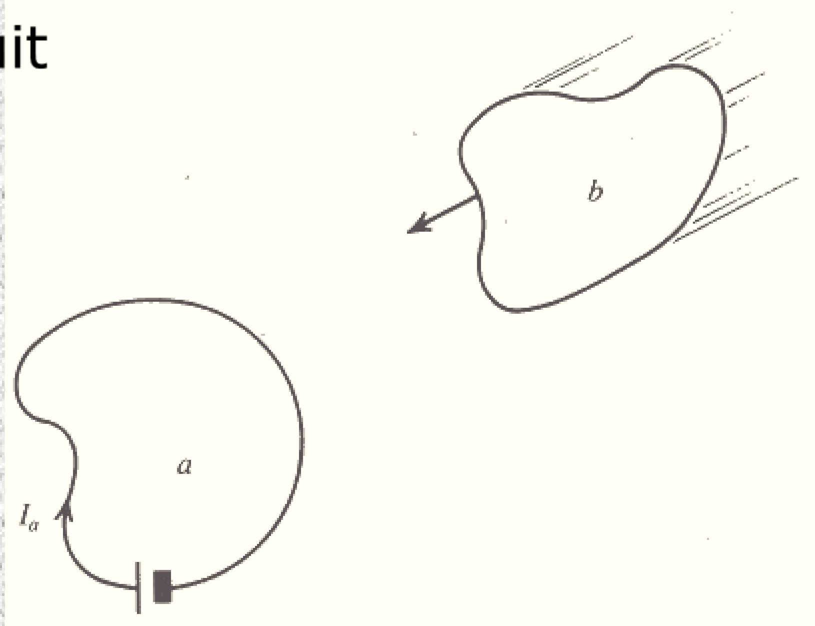
(2) $d\emptyset/dt$

$$\mathcal{V} = -\frac{d\Phi}{dt} = -\frac{d}{dt} abB \cos\omega t = abB\omega \sin\omega t. \qquad (23\text{-}26)$$



Faraday's Unduction Law for Time-dependent B's.

- Imagine two closed and rigit circuit as in figure. The active circuit *a* is stationary and the passive circuit *b* moves in some arbitrary way. The current I_a constant



- The electromotance induced in *b*

$$\mathcal{V} = \oint_b (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt}, \quad (23\text{-}27)$$

- Where ∅ is the magnetic flux linking *b*
- The electromotance induced in a rigid and stationary circuit clying in time-varying magnetic field is

$$\mathcal{V} = \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{\mathcal{A}} (\nabla \times \boldsymbol{E}) \cdot d\mathcal{A} = -\frac{d\Phi}{dt} = -\int_{\mathcal{A}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\mathcal{A}. \quad (23\text{-}29)$$

- $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$. This is another Maxwell's equation

Electric field Strength E Expressed in Terms of V and A

- An arbitrary rigit and stationary closed circuit lies in a time-dependent **B.** Then

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\int_{\mathscr{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathscr{A}}, \qquad (23\text{-}41)$$

- We can replace the surface integral on the right by the line integral of the vector potantial **A** around *C*:

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\oint_C \mathbf{A} \cdot d\boldsymbol{l} = -\oint_C \frac{\partial \mathbf{A}}{\partial t} \cdot d\boldsymbol{l}. \qquad (23\text{-}42)$$

- Thus $$\oint_C \left(\boldsymbol{E} + \frac{\partial \mathbf{A}}{\partial t}\right) \cdot d\boldsymbol{l} = 0,$$
- The expression enclosed in the paranthesis is equal to the gradient of some function:

$$\boldsymbol{E} + \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla} V, \qquad (23\text{-}44)$$

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \qquad (23\text{-}45)$$

- The faraday induction law relates the space derivatives of **E** to the time derivative of **B** at given point

The E, ∇V, $\partial A/\partial t$ and vxB Fields

- In any given inertial frame, say *S*, the equation

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

always applies.

- If a charge *Q* moves at a velocity *v* with respect to *S*, then for an observer on *S* the force is

$$F = Q(E + v \times B) = Q\left(-\nabla V - \frac{\partial A}{\partial t} + v \times B\right). \qquad (23\text{-}59)$$

- All the variables are measured with respect to the same reference frame *S.*

Six Key Equations

- It is useful at this stage to group the following six equations:

(G) $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad (23\text{-}46)$$

(G) $$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{\mathcal{A}}, \qquad (23\text{-}29)$$

(G) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad (23\text{-}30)$$

(G) $$\mathbf{B} = \nabla \times \mathbf{A}, \qquad (23\text{-}46)$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\mathcal{A}} \mathbf{J} \cdot d\boldsymbol{\mathcal{A}},$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

- The four equation preceded by (G) are general, while the other two apply only to slowing varying fields.
- In each equation all the terms concern with same reference frame

Magnetic Energy Density Expressed in Terms of J and A

- Since Flux linkage is $\Lambda = LI$, we can rewrite the time derivative of magnetic energy as follows;

$$\frac{d\mathscr{E}_m}{dt} = \frac{1}{2}\left(I\frac{d\Lambda}{dt} + \Lambda\frac{dI}{dt}\right) = I\frac{d\Lambda}{dt} = I\oint_C \frac{\partial \mathbf{A}}{\partial t} \cdot d\boldsymbol{l}$$

$$= \int_{\mathscr{A}} \boldsymbol{J} \cdot d\boldsymbol{\mathscr{A}} \oint_C \frac{\partial \mathbf{A}}{\partial t} \cdot d\boldsymbol{l} \qquad (26\text{-}14)$$

$$= \int_v \boldsymbol{J} \cdot \frac{\partial \mathbf{A}}{\partial t}\, dv, \qquad (26\text{-}15)$$

- Now consider the identity

$$\frac{d}{dt}\int_v \boldsymbol{J} \cdot \mathbf{A}\, dv = \int_v \boldsymbol{J} \cdot \frac{\partial \mathbf{A}}{dt}\, dv + \int_v \frac{\partial \boldsymbol{J}}{\partial t} \cdot \mathbf{A}\, dv. \qquad (26\text{-}16)$$

- Therefore

$$\frac{d\mathscr{E}_m}{dt} = \frac{1}{2}\frac{d}{dt}\int_v \boldsymbol{J} \cdot \mathbf{A}\, dv \qquad (26\text{-}17)$$

$$\mathscr{E}_m = \frac{1}{2}\int_v \boldsymbol{J} \cdot \mathbf{A}\, dv, \qquad (26\text{-}18)$$

- The magnetic energy density at a point is $\mathscr{E}'_m = \frac{1}{2}\boldsymbol{J} \cdot \mathbf{A}.$

Magnetic Energy Density Expressed in Terms of H and B

- From ampere's law

$$I = \oint_{C'} H\, dl, \qquad (26\text{-}20)$$

- Where *C'* is any line of **H**. Then

$$\Lambda = \Phi = \int_{\mathcal{A}} \boldsymbol{B} \cdot d\mathcal{A} \qquad (26\text{-}21)$$

$$\mathscr{E}_m = \tfrac{1}{2} I \Lambda = \frac{1}{2} \oint_{C'} H\, dl \int_{\mathcal{A}} B\, d\mathcal{A}. \qquad (26\text{-}22)$$

- Since the fields extends to infinity, this double integral is the volume integral of **H.B** over all space and

$$\mathscr{E}_m = \frac{1}{2} \int_{\infty} \boldsymbol{H} \cdot \boldsymbol{B}\, dv. \qquad (26\text{-}23)$$

- The magnetic energy density in nonferromagnetic madia is thus

$$\mathscr{E}'_m = \frac{\boldsymbol{H} \cdot \boldsymbol{B}}{2} = \frac{B^2}{2\mu} = \frac{\mu H^2}{2}. \qquad (26\text{-}24)$$