EEE321 Electromagnetic Fileds and Waves

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(5^{th} Week)

Outline

- Lorentz Force
- Magnetic Flux Density **B**. Biot-Savart Law
- Divergence of **B**
- Vector Potential **A**.
- Magnetic Dipole Moment of an Arbitrary Current Distribution
- Line integral of **A.*dl***
- Laplasian of **A**
- Divergence of **A**
- Ampere’s Circuital Law
- Laplasian of **B**

Lorentz Force

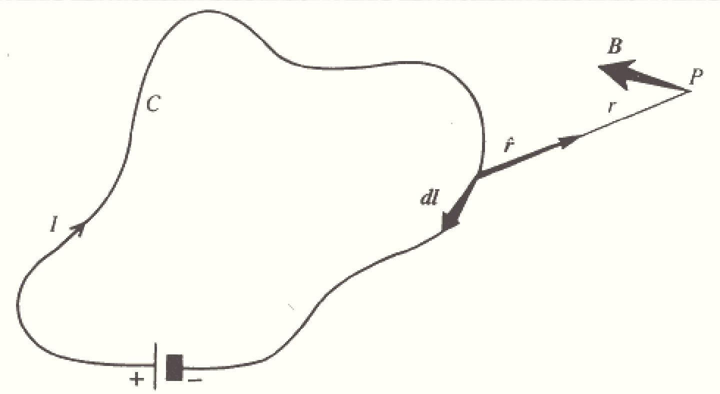
- Imagine a set of charges moving around space. At any point ***r*** in the space and at any time *t* there exits an electric field strength **E**(***r***,*t)* and a magnetic flux density **B**(***r***,*t)* taht are defined as follows. If a charge *Q* moves at velocity ***v*** at (***r***,*t)* in this field, then it suffers a Lorentz force

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \qquad (18\text{-}1)$$

- The Electric force is proportional to Q but independent of ***v***, while the magnetic force is orthogonal to both ***v*** and **B**

The Magnetic Flux Density B and Biot-Savart Law

- If the electric circuit carrying a steady current I, there exists a field at a point P in the space as;



$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}. \qquad (18\text{-}5)$$

- This is called Biot-Savart Law. The unit vector ***r*** points from the source to the point of observation.
- The unit of magnetic flux density is *tesla*.

$$\text{Tesla} = \frac{\text{volt}}{\text{meter}}\,\frac{\text{second}}{\text{meter}} = \frac{\text{weber}}{\text{meter}^2}. \qquad (18\text{-}6)$$

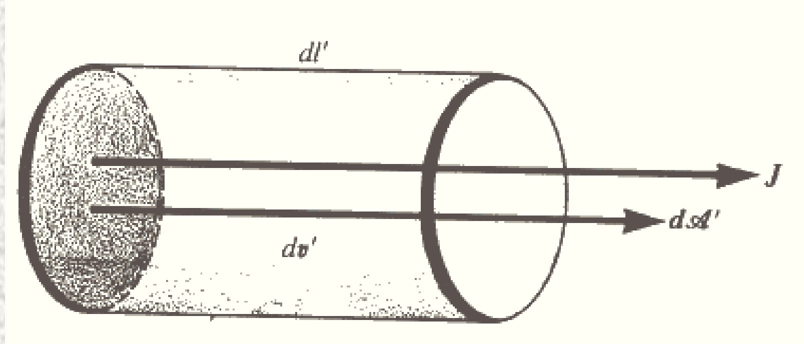
- By definition

$$\mu_0 = 4\pi \times 10^{-7}\ \text{weber/ampere-meter}. \qquad (18\text{-}7)$$

- In this circuit we assumed a current flowing through a thin wire

- If the current flows over a finite volume

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}}{r^2} dv', \qquad (18\text{-}8)$$

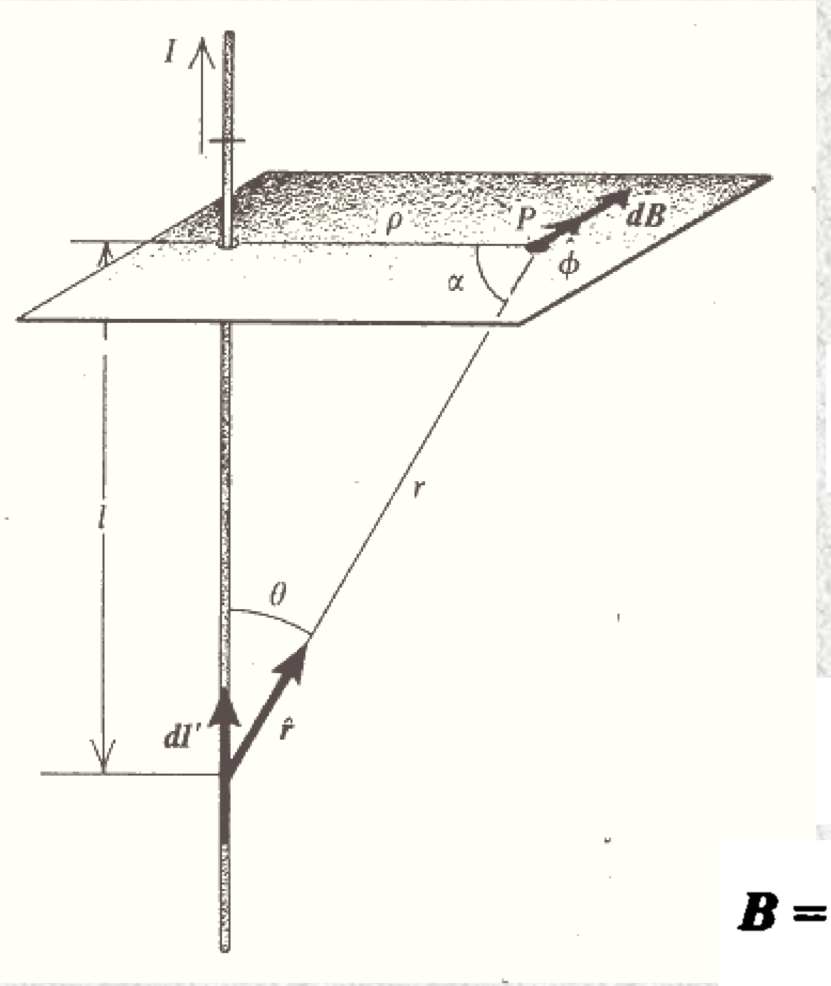


- in which *v′* is any volume enclosing all the current and *r* is the distance between the elements of volüme *dv* anf the point *P*.
- Lines of **B** points everywhere in the direction of **B**.
- Magnetic flux through a surface of area *A* is

$$\Phi = \int_{\mathcal{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathcal{A}} \text{ webers.} \qquad (18\text{-}9)$$

- This surface is usualy open. If it is closed ϕ=0.

Example-A Long Straight Wire



- An element *dl′* of a long straight wire carrying current *I* as shown in figure.
- At a point *P(r,θ,φ),* a magnetic flux density

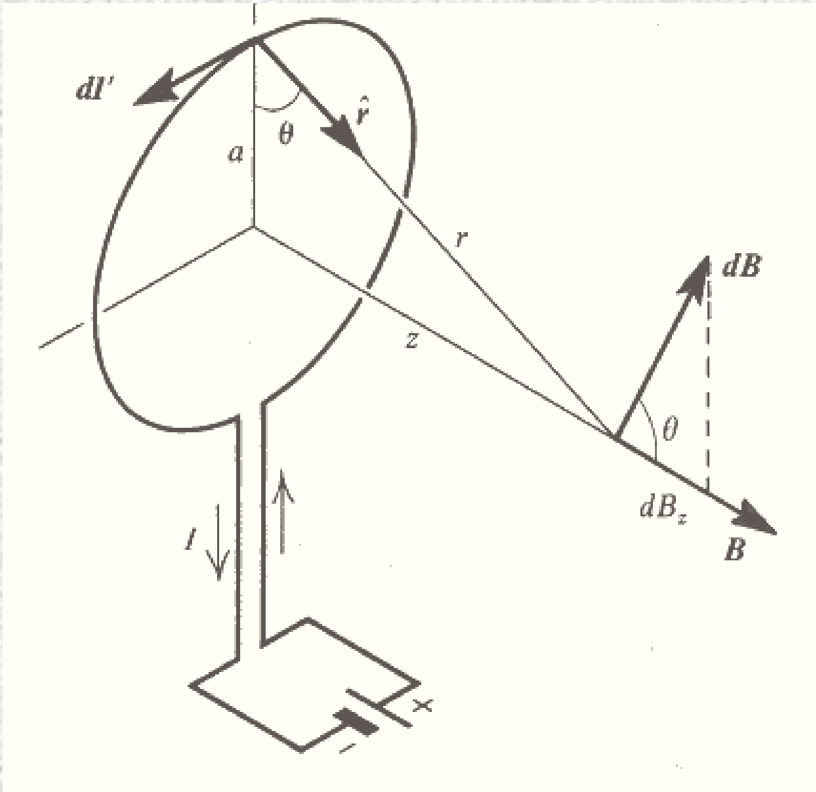
$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl'\sin\theta}{r^2}\hat{\phi} = \frac{\mu_0}{4\pi}\frac{I\,dl'\cos\alpha}{r^2}\hat{\phi}. \qquad (18\text{-}10)$$

- The relative orientations of *I* and **B** satisfy right-hand screw rule

$$l = \rho\tan\alpha, \qquad dl' = \frac{r\,d\alpha}{\cos\alpha} = \frac{r^2\,d\alpha}{\rho}. \qquad (18\text{-}11)$$

$$B = \frac{\mu_0 I}{4\pi\rho}\int_{-\pi/2}^{+\pi/2}\cos\alpha\,d\alpha\,\hat{\phi} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}. \qquad (18\text{-}12)$$

Example-The Circular Loop



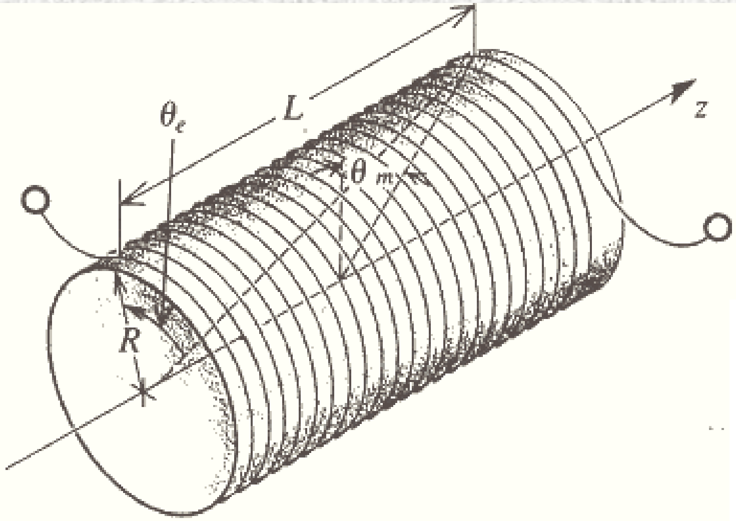
- To calculate the value of **B** on the axis of a circular loop of redius *a* shown in the figüre.
- By symmety, the total **B** points along the axis and

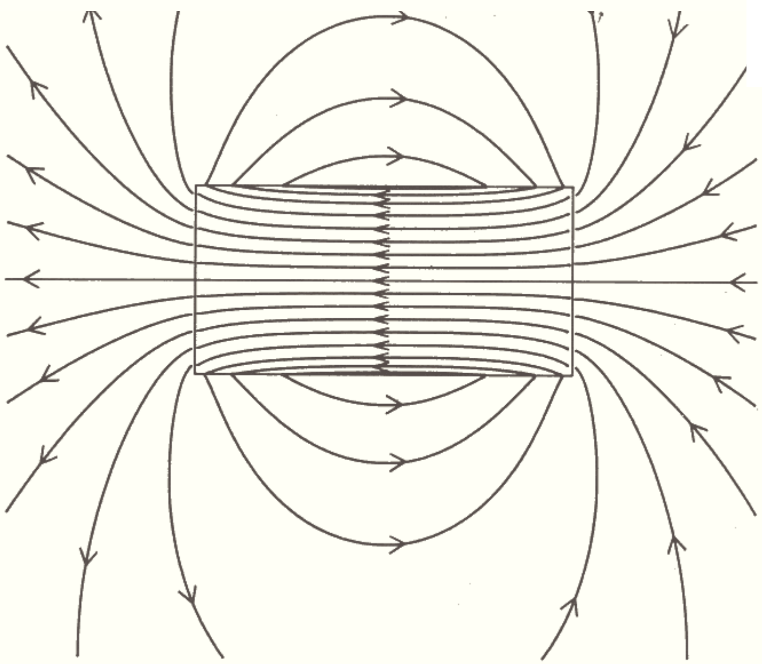
$$dB_z = \frac{\mu_0}{4\pi} \frac{I\,dl'}{r^2} \cos\theta, \qquad (18\text{-}13)$$

$$B_z = \frac{\mu_0}{4\pi} \frac{2\pi a I}{r^2} \cos\theta = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}. \qquad (18\text{-}14)$$

- Along the axisi $B=\mu_0 I/2a$ at $z=0$ and falls off as $1/z^3$ for $z^2 >> a^2$

Example-The Selenoid





- The selenoid is close-wound of lenght *L* with *N′* turns per meter, and its Radius is *R*. At the center

$$B = \frac{\mu_0}{2}\int_{-L/2}^{+L/2} \frac{R^2 N' I \, dz}{(R^2 + z^2)^{3/2}} \qquad (18\text{-}15)$$

$$= \frac{\mu_0}{2} N' I \frac{L}{(R^2 + L^2/4)^{1/2}} = \mu_0 N' I \sin\theta_m. \qquad (18\text{-}16)$$

- See figüre for the definition of θm and θe. At one end, again on the axis

$$B = \frac{\mu_0 N' I \sin\theta_e}{2}. \qquad (18\text{-}17)$$

- Inside a long selenoid $B \approx \mu_o N' I$

The Divergence of B

- Assuming that magnetic monopoles do not exist, all magnetic fileds results from electric current and the lines of **B** for each element of current are circles. Thus the net outward flux of **B** through any closed surface is zero

$$\int_{\mathscr{A}} \mathbf{B} \cdot d\mathscr{A} = 0. \qquad (18\text{-}18)$$

- Applying the divergence theorem, it follows that

$$\nabla \cdot \mathbf{B} = 0. \qquad (18\text{-}19)$$

- These are alternative forms of one of Maxwell's equations

The Vector Potential A

- We just seen that

$$\nabla \cdot \boldsymbol{B} = 0. \qquad (18\text{-}19)$$

- It is convenient to set

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad (18\text{-}20)$$

- Where **A** is the vector potential, as opposed to *V*, which is the scalar potential
- We can deduce the integral for **A**, starting from Biot-Savart law.

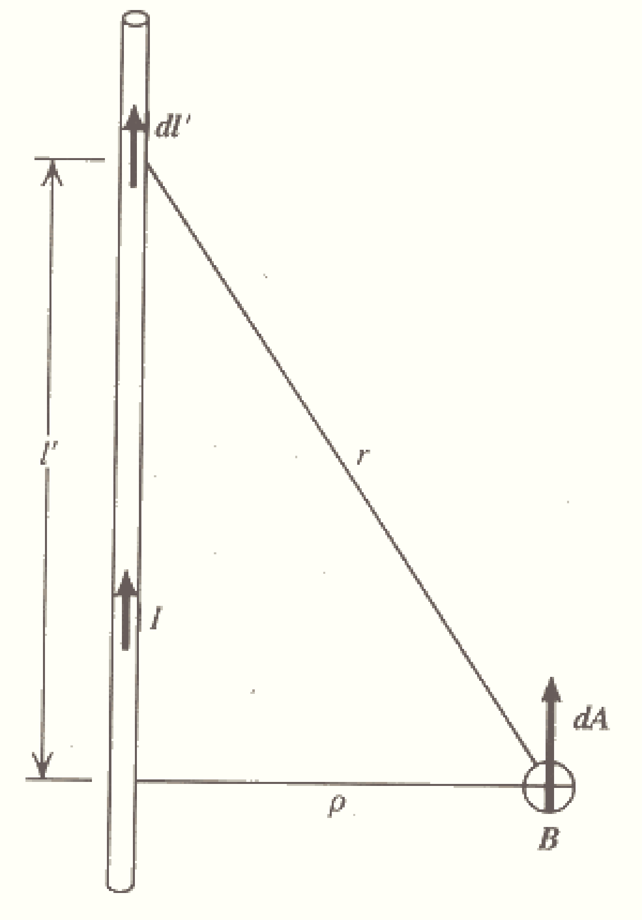
$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}}{r^2}\, dv' = \frac{\mu_0}{4\pi}\int_{v'} \left(\nabla \frac{1}{r}\right) \times \boldsymbol{J}\, dv', \qquad (18\text{-}22)$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_{v'} \left(\nabla \times \frac{\boldsymbol{J}}{r}\right) dv' = \nabla \times \left(\frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J}}{r}\, dv'\right), \qquad (18\text{-}24)$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J}}{r}\, dv'. \qquad (18\text{-}25)$$

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi}\int_{C} \frac{\boldsymbol{dl}'}{r}, \qquad (18\text{-}26)$$

Example-A and B Near a Long, Straight Wire



- We first calculate **A** and then deduce **B.** At a distance ρ

$$dA = \frac{\mu_0}{4\pi}\frac{I\,dl'}{r}. \qquad (18\text{-}27)$$

$$A = \frac{\mu_0 I}{4\pi}\int_{-L/2}^{L/2}\frac{dl'}{(\rho^2 + l'^2)^{1/2}} = \frac{\mu_0 I}{2\pi}\left(\ln\frac{l' + (\rho^2 + l'^2)^{1/2}}{\rho}\right)_0^{L/2} \qquad (18\text{-}28)$$

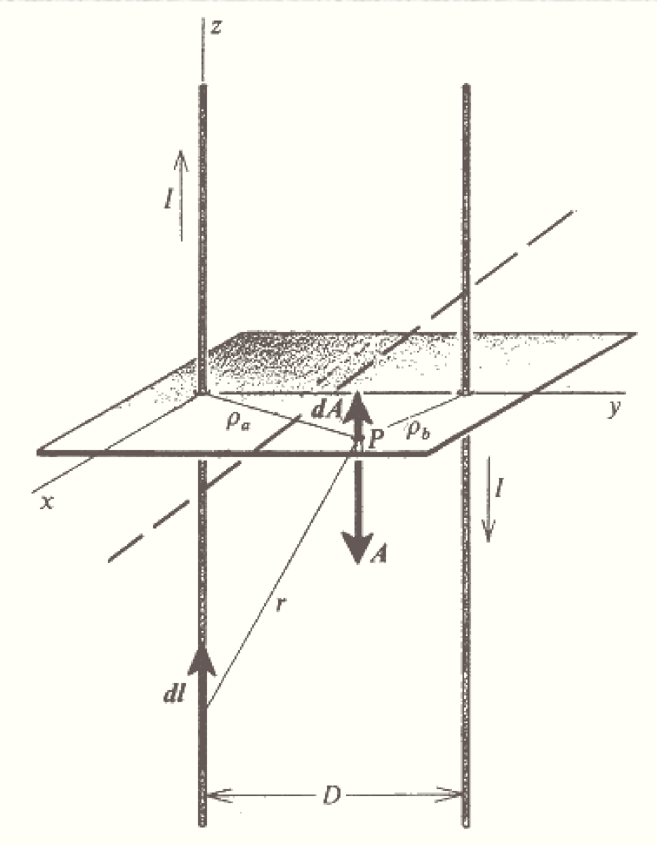
$$= \frac{\mu_0 I}{2\pi}\ln\frac{(L/2)[1 + (1 + 4\rho^2/L^2)^{1/2}]}{\rho} \approx \frac{\mu_0 I}{2\pi}\ln\frac{L}{\rho} \qquad (4\rho^2 \ll L^2)$$

$$\approx \frac{\mu_0 I}{2\pi}\ln\frac{\mathcal{R}}{\rho} \qquad (4\rho^2 \ll L^2) \qquad (18\text{-}29)$$

- To calculate $\mathbf{B} = \nabla X A$, we use cylindirical coordinates, keeping in mind that **A** is paralel to z-axis and independent of both φ and z

$$\boldsymbol{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}} \qquad (4\rho^2 \ll L^2), \qquad (18\text{-}30)$$

Example-Pair of Long Parallel Currents



- Figure shows two long parallel wires separated by a distance *D* and currying equal current *I* in oposite directions
- To calculate **A**, we use **A** of a single wire and add the two vector potentials

$$A = \frac{\mu_0 I}{2\pi}\left(\ln\frac{L}{\rho_a} - \ln\frac{L}{\rho_b}\right) = \frac{\mu_0 I}{2\pi}\ln\frac{\rho_b}{\rho_a} = \frac{\mu_0 I}{4\pi}\ln\frac{x^2 + (D-y)^2}{x^2 + y^2}. \qquad (18\text{-}31)$$

- Then

$$B_x = \frac{\partial A}{\partial y} = -\frac{\mu_0 I}{2\pi}\left(\frac{D-y}{\rho_b^2} + \frac{y}{\rho_a^2}\right), \qquad (18\text{-}32)$$

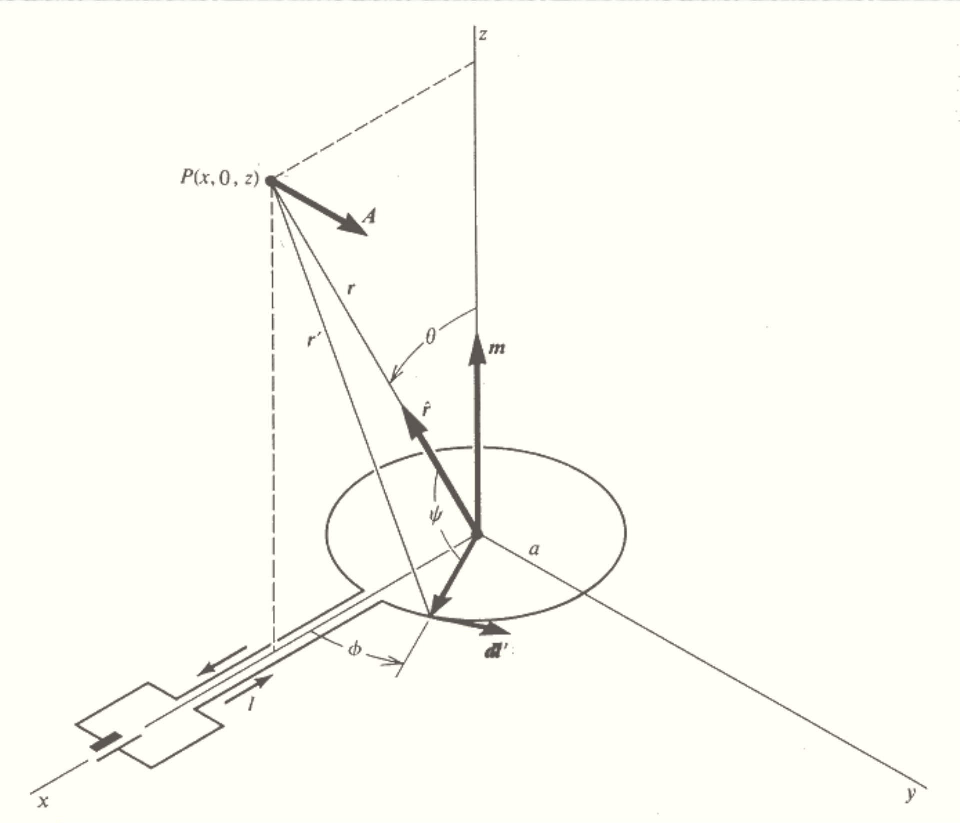
$$B_y = -\frac{\partial A}{\partial x} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{\rho_a^2} - \frac{1}{\rho_b^2}\right)x, \qquad (18\text{-}33)$$

$$B_z = 0. \qquad (18\text{-}34)$$

- Along the line midway between wires

$$B_x = -\frac{2\mu_0 I}{\pi D}, \qquad B_y = 0, \qquad B_z = 0. \qquad (18\text{-}35)$$

Example-A of Magnetic dipole



- Magnetic dipole is a loop of wire carrying a current *I* as in figure.
- We will find **A** then deduce **B**.
- At the point *P* in the figüre

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{l}'}{r}. \qquad (18\text{-}36)$$

- By symetry, **A** is azimuthal, and

$$A = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a\, d\phi \cos\phi}{r'}. \qquad (18\text{-}37)$$

$$r'^2 = r^2 + a^2 - 2ar\cos\psi. \qquad (18\text{-}38)$$

$$x\cos\phi = r\cos\psi \qquad (18\text{-}39)$$

$$r'^2 = r^2 + a^2 - 2ax\cos\phi, \qquad (18\text{-}40)$$

$$r' = r\left\{1 + \left[\frac{a^2}{r^2} - 2\frac{a}{r}\left(\frac{x}{r}\cos\phi\right)\right]\right\}^{1/2} = r\{1 + [\ \ ]\}^{1/2}. \qquad (18\text{-}41)$$

- Since, we are interested in the field only at points where *r>>a*, we expand *1/r'* as an infinite series and disregard terms involving higer Powers of *a/r.* Thus

$$\frac{1}{r'}=\frac{1}{r}\left\{1-\frac{1}{2}[\ \ ]+\frac{3}{8}[\ \ ]^2-\cdots\right\}. \qquad (18\text{-}42)$$

- Setting

$$\left(\frac{x}{r}\cos\phi\right)=(\ \ ), \qquad (18\text{-}43)$$

- We find that

$$\frac{1}{r'}=\frac{1}{r}\left\{1-\frac{1}{2}\left[\frac{a^2}{r^2}-2\frac{a}{r}(\ \ )\right]+\frac{3}{8}\left[\frac{a^4}{r^4}-4\frac{a^3}{r^3}(\ \ )+4\frac{a^2}{r^2}(\ \ )^2\right]-\cdots\right\}. \qquad (18\text{-}44)$$

- Disgarding all terms containing the thirdand hiher power of *a/r*

$$\frac{1}{r'}=\frac{1}{r}\left\{1+\frac{a}{r}(\ \ )-\left[\frac{1}{2}-\frac{3}{2}(\ \ )^2\right]\frac{a^2}{r^2}\right\}. \qquad (18\text{-}45)$$

- Finaly

$$A = \frac{\mu_0 I a}{4\pi r} \int_0^{2\pi} \left[1 + \frac{a}{r}\left(\frac{x}{r}\cos\phi\right) - \left(\frac{1}{2} - \frac{3}{2}\frac{x^2}{r^2}\cos^2\phi\right)\frac{a^2}{r^2}\right] \cos\phi \, d\phi. \qquad (18\text{-}46)$$

$$A = \frac{\mu_0 I a^2 x}{4r^3} = \frac{\mu_0 I a^2 \sin\theta}{4r^2} \qquad (r^3 \gg a^3). \qquad (18\text{-}47)$$

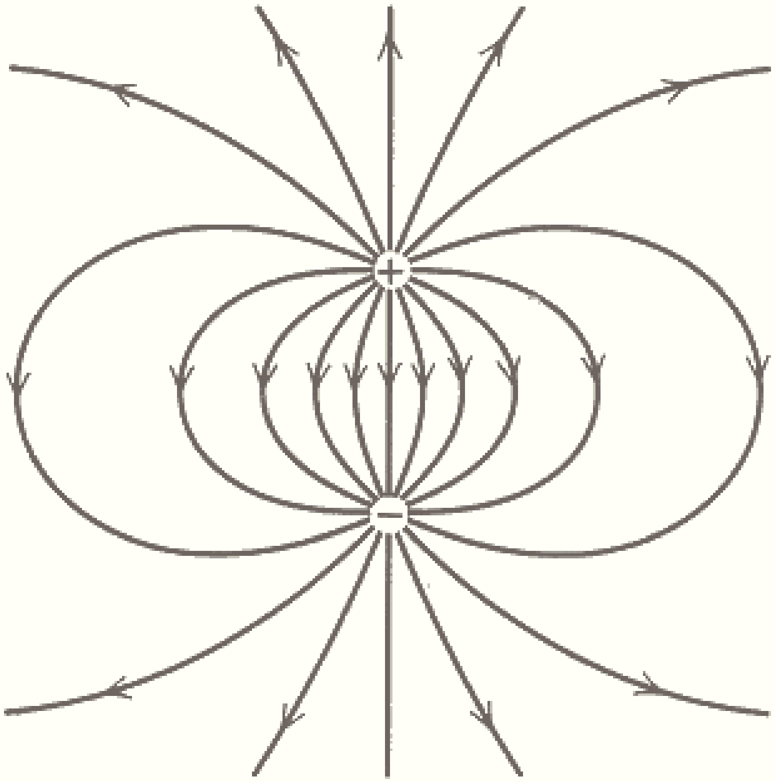
- By definition

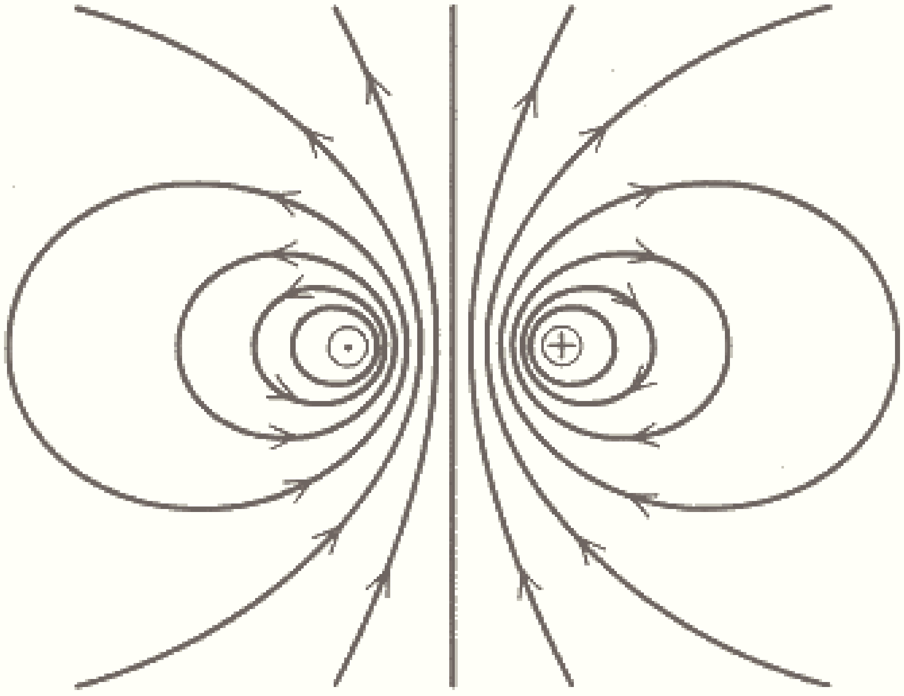
$$\mathbf{m} = \pi a^2 I \hat{\mathbf{z}} \qquad (18\text{-}48)$$

- İs the magnetic dipole moment of the loop. If there are N turns then **m** is *N* times larger.
- Since A is azimuthal

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \qquad (r^3 \gg a^3). \qquad (18\text{-}49)$$

- The fields (a) of an electic dipole and (b) of a magnetic dipole in the imminidate vicinity of the dipoles


(a)


(b)

B in the Field of a Magnetic Dipole

- The value of $\boldsymbol{B} = \nabla\ x\ \boldsymbol{A}$ follows immidiately

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right). \tag{18-50}$$

- The analogy with the field of electric dipole is obvious.

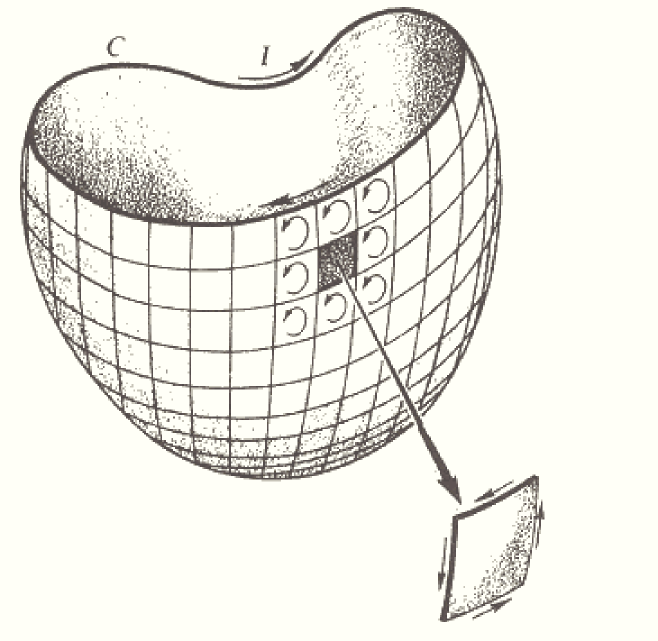
Magnetic Dipole Moment of an Arbitrary Current Distribution

- Assume first a plane loop of arbitrary shape carrying a current *I*. Then we set

  $$\mathbf{m} = \mathscr{A}I\hat{\mathbf{z}}, \qquad (18\text{-}51)$$

  where *A* is the area of the loop, and **z** is normal to the loop, and satisfy right-hand screw. We can write

  $$\mathbf{m} = \tfrac{1}{2}I\oint_C \mathbf{r} \times d\mathbf{l}', \qquad (18\text{-}52)$$



- Magnetic dipole moment of *C* is the vector sum of the magnetic dipoles of the individual cells. Thus

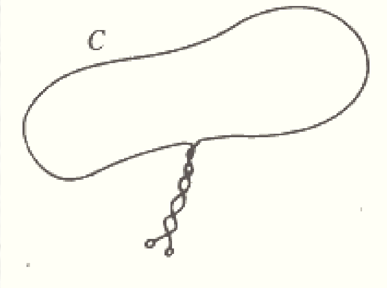
  $$\mathbf{m} = \sum \tfrac{1}{2}I\oint_{\text{cell}} \mathbf{r} \times d\mathbf{l}' = \tfrac{1}{2}I\oint_C \mathbf{r} \times d\mathbf{l}', \qquad (18\text{-}53)$$

- An arbitrary current distribution

  $$\mathbf{m} = \tfrac{1}{2}\int_{v'} \mathbf{r} \times \mathbf{J}\, dv'. \qquad (18\text{-}54)$$

Line integral of A.*dl*

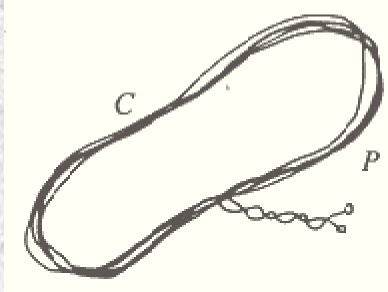
- Consider a simple closed curve as in (a)
- The line integral of **A.*dl*** around *C* is equal



(a)

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_{\mathcal{A}} (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\mathcal{A}} = \int_{\mathcal{A}} \mathbf{B} \cdot d\boldsymbol{\mathcal{A}} = \Phi, \qquad (19\text{-}1)$$

- Now suppose the coil has N turn turns as in (b)



(b)

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = N \int_{\mathcal{A}} \mathbf{B} \cdot d\boldsymbol{\mathcal{A}} = N\Phi = \Lambda, \qquad (19\text{-}2)$$

- Where Λ is the ***flux linkage*** (unit weber turn)
- For (c)

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_{\mathcal{A}} \mathbf{B} \cdot d\boldsymbol{\mathcal{A}} = \Lambda, \qquad (19\text{-}3)$$

- Although the surface bounded by *C* is very difficult to find, the flux linkage Λ is easly measurable

Laplacian of A

- As we will recall from electric field that

$$V = \frac{1}{4\pi\epsilon_0}\int_{v'}\frac{\rho}{r}\,dv', \qquad \nabla^2 V = -\frac{\rho}{\epsilon_0}. \qquad (19\text{-}7)$$

- There exists an anologous pair of equations for vector potential **A**.
- We have already found that

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{v'}\frac{\mathbf{J}}{r}\,dv', \qquad (19\text{-}8)$$

- Where *v′* is any volüme enclosing all the currents. The *x* component of this equation is

$$A_x = \frac{\mu_0}{4\pi}\int_{v'}\frac{J_x}{r}\,dv'. \qquad (19\text{-}9)$$

- By analogy $$\nabla^2 A_x = -\mu_0 J_x. \qquad (19\text{-}10)$$
- Then $$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \qquad (19\text{-}11)$$

Divengence of A

- To find divergence of **A**, first

$$\nabla \cdot \mathbf{A} = \nabla \cdot \frac{\mu_0}{4\pi} \int_{v'} \frac{\mathbf{J}}{r} dv' = \frac{\mu_0}{4\pi} \int_{v'} \nabla \cdot \left(\frac{\mathbf{J}}{r}\right) dv', \qquad (19\text{-}12)$$

$$\nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \int_{v'} \left(\nabla \frac{1}{r}\right) \cdot \mathbf{J}\, dv' = -\frac{\mu_0}{4\pi} \int_{v'} \left(\nabla' \frac{1}{r}\right) \cdot \mathbf{J}\, dv' \qquad (19\text{-}13)$$

$$= \frac{\mu_0}{4\pi} \int_{v'} \left(-\nabla' \cdot \frac{\mathbf{J}}{r} + \frac{\nabla' \cdot \mathbf{J}}{r}\right) dv'. \qquad (19\text{-}14)$$

- In the time-independant field $\frac{\partial \rho}{\partial t} = 0$ and the conservation of charge $\nabla' . \boldsymbol{J} = \mathbf{0}$. Therefore

$$\nabla \cdot \mathbf{A} = -\frac{\mu_0}{4\pi} \int_{v'} \nabla' \cdot \frac{\mathbf{J}}{r} dv' = -\frac{\mu_0}{4\pi} \int_{\mathcal{A}'} \frac{\mathbf{J}}{r} \cdot d\boldsymbol{\mathcal{A}}' \equiv 0, \qquad (19\text{-}15)$$

- As a result divergence of **A** is zero.

Curl of B

- From the definitions given before

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \qquad (19\text{-}16)$$

- Since Divergenc eof **A** is zero.

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \qquad (19\text{-}17)$$

Ampere's Circuital Law

- The line integral of **B.*dl*** around a closed curve C is important

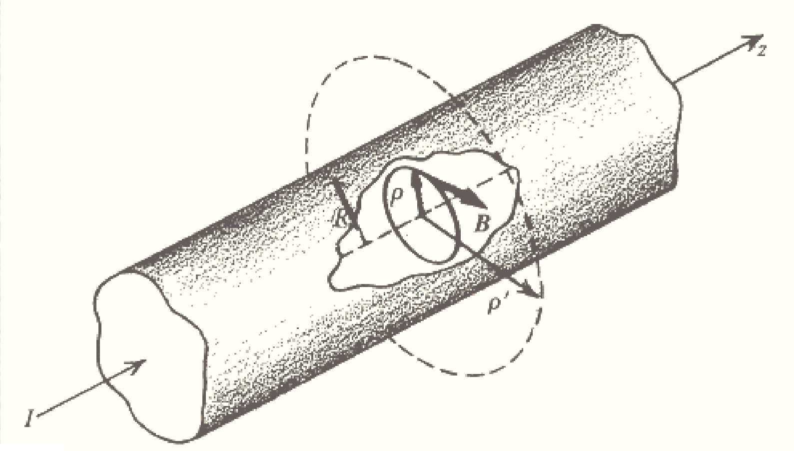
$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \int_{\mathcal{A}} (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{\mathcal{A}} = \mu_0 \int_{\mathcal{A}} \boldsymbol{J} \cdot d\boldsymbol{\mathcal{A}} = \mu_0 I. \qquad (19\text{-}18)$$

- This is called Ampere's Circuital Law. The line integral of **B.*dl*** around a closed curve C is equal to μ_o times the current linking *C*.
- This law is analogous to Gaus's law, which is calculated to find **E**.

Example-Long cylindrical Conductor

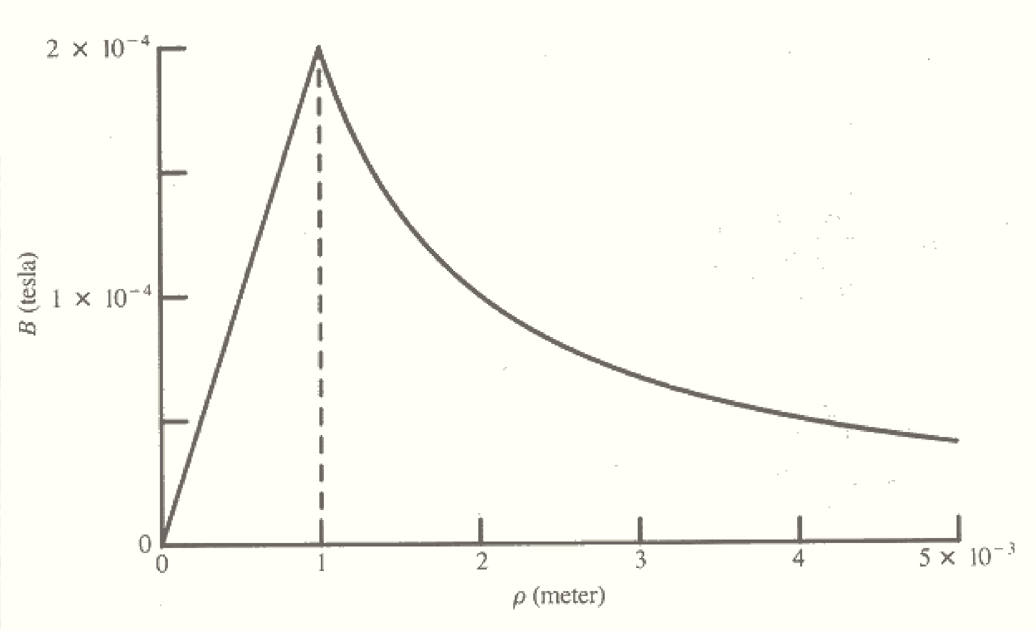
- Outside the conductor, **B** is azimuthal and independent of φ. Then

$$B = \frac{\mu_0 I}{2\pi\rho'}. \qquad (19\text{-}19)$$



- Inside the vonductor, for the circuital path of Radius ρ,

$$B = \mu_0 \frac{[I/(\pi R^2)]\pi\rho^2}{2\pi\rho} = \frac{\mu_0 I\rho}{2\pi R^2}.$$



Laplacian of B

- We can deduce the value of the laplacian of **B** from that of the laplacian of **A**. Since

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \qquad (19\text{-}22)$$

- Then

$$\nabla \times (\nabla^2 \mathbf{A}) = -\mu_0 \nabla \times \mathbf{J}. \qquad (19\text{-}23)$$

- The curl of a laplacian is equal to the laplacian of a curl and thus

$$\nabla^2 (\nabla \times \mathbf{A}) = -\mu_0 \nabla \times \mathbf{J}. \qquad (19\text{-}24)$$

$$\nabla^2 \mathbf{B} = -\mu_0 \nabla \times \mathbf{J}, \qquad (19\text{-}25)$$