EEE321 Electromagnetic Fileds and Waves

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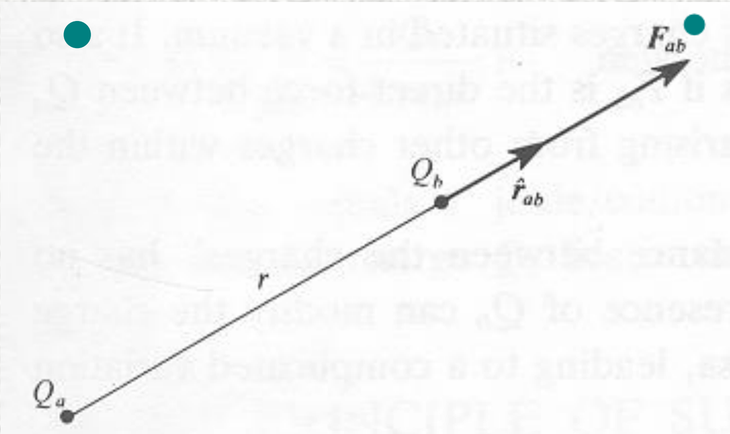
(2nd Week)

Outline

- Coulomb’s Law
- The Electrik Fields Strength **E**
- The Principles of Superposition
- The Electric Potantial
- The Equations of Poisson and Laplace
- The Law of Conservation of Electric
- Conduction

Coulomb's Law

- 

Experiments Show that the force exerted by a stationary poit charge Q_a on the stationary point charge Q_b stuated a distance *r* is given by

$$F_{ab} = \frac{Q_a Q_b}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}_{ab}, \qquad (3\text{-}1)$$

The unit vector $\hat{\mathbf{r}}_{ab}$ points from Q_a to Q_b. This is ***Coulumb's Law****. Charges are measured in coulumbs, the force in Newton and the distance in meters*

- *The constant ϵ_o is the permittivity of free space*

$$\epsilon_0 = 8.854187817 \times 10^{-12} \text{ farad/meter.} \qquad (3\text{-}2)$$

- *Substituting the value of ϵ_o, we found that*

$$F_{ab} \approx 9 \times 10^9 \frac{Q_a Q_b}{r^2} \quad \text{newtons,} \qquad (3\text{-}3)$$

The Electrik Fields Strength E

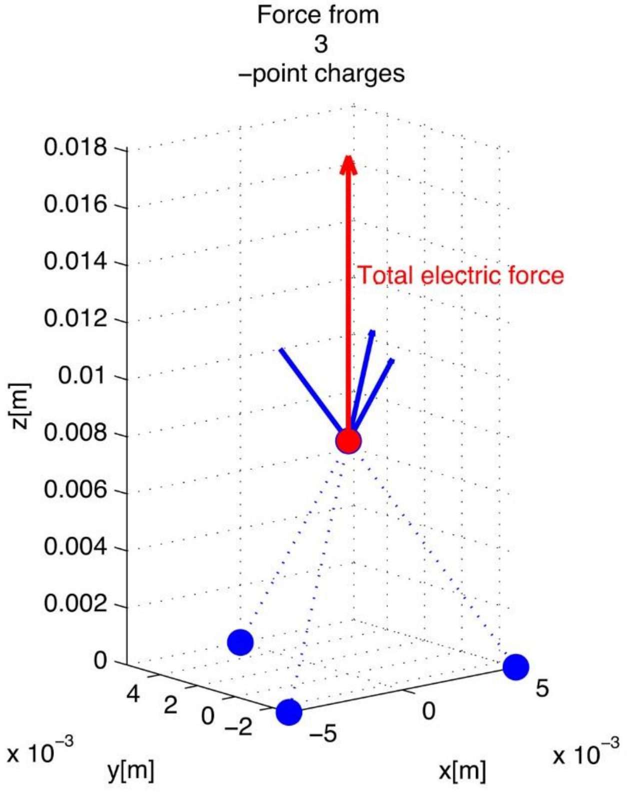
- Electric Field Strength **E** is defined at a point that the force excerted on a unit test charge situated at that point

$$\mathbf{E}_a = \frac{\mathbf{F}_{ab}}{Q_b} = \frac{Q_a}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \qquad \text{newtons/coulomb, or volts/meter,} \qquad (3\text{-}5)$$

- The force between two electric charges Q_a and Q_b results from the interaction of Q_b with the field of Q_a at the position of Q_b, or vice versa

The Principles of Superposition

- If there are several charges, each one imposes its own field, and the resultant **E** is simply the vector sum of all the individual **E**'s. This is the **Principle of Superposition.**



- For the continuous distribution of charge, the electric field strength **E**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\rho\hat{\mathbf{r}}}{r^2}\,dv', \qquad (3\text{-}6)$$

The Electric Potential *V* and the Curl of E-1

- Consider a test charge Q′ that can move about in an electric field. The total energy required is

$$\mathscr{E} = -\int_A^B \mathbf{E}Q' \cdot d\mathbf{l}. \qquad (3\text{-}7)$$

- If the path is closed, the total work done on Q′ is

$$\mathscr{E} = -\oint \mathbf{E}Q' \cdot d\mathbf{l}. \qquad (3\text{-}8)$$

- Let us consider that the electric field produced by a single stationary point charge Q, then

$$\oint \mathbf{E}Q' \cdot d\mathbf{l} = \frac{QQ'}{4\pi\epsilon_0}\oint \frac{\hat{\mathbf{r}} \cdot d\mathbf{l}}{r^2}. \qquad (3\text{-}9)$$

- This line integral is zero and the net work is fixed, is zero. Thus for any distribution of fixed charges,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0. \qquad (3\text{-}10)$$

- An electrostatic field is conservative

The Electric Potential *V* at a Point

- It is usualy convenient to choose the potential *V* at infinity to be zero, then at point P

$$V = \int_{P}^{\infty} \mathbf{E} \cdot d\mathbf{l}. \tag{3-16}$$

- If the field is that of a single point charge, the potential is

$$V = \int_{r}^{\infty} \frac{Q}{4\pi\epsilon_0} \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 r}. \tag{3-17}$$

- For any charge distribution of density ρ

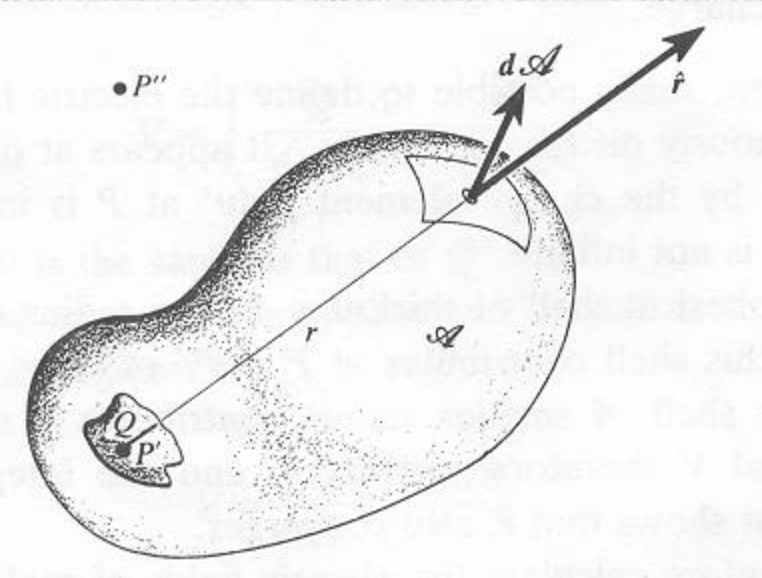
$$V = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho\, dv'}{r}, \tag{3-18}$$

- The principle of superposition applies to *V* as to **E**

Equipotantial Surfaces and Lines of E

- The set of all points in the space that are at a given potantial defines an equipotantial surface.
- The equipotantial surface about a point charge are concentric spheres.
- Since $\boldsymbol{E} = -\nabla V$, **E** is everywhere normal to the equipotantial surface

Gauss's Law



- Consider that a finite valume *v* bounded by a surface *A* encloses a charge Q. The flux of **E** through the element of area ***dA*** *is*

$$\mathbf{E} \cdot d\mathcal{A} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}} \cdot d\mathcal{A}}{r^2}. \qquad (3\text{-}19)$$

- To find the outward flux of **E**, we integrate over the area *A*

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathcal{A} = \frac{Q}{\epsilon_0}. \qquad (3\text{-}21)$$

- This is Gauss's Law. If the charge occupies a finite valume, then

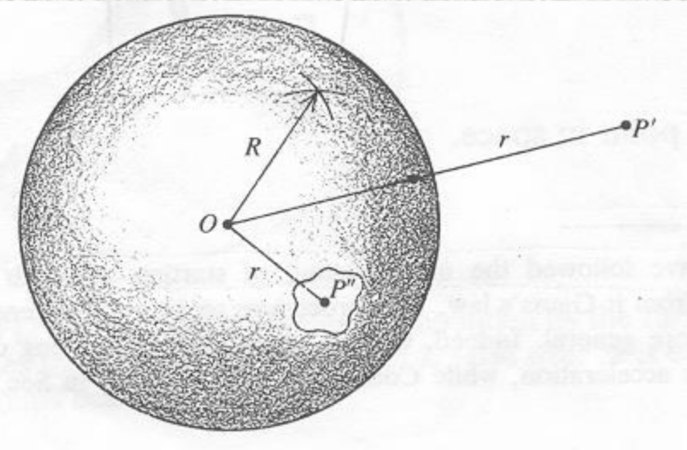
$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathcal{A} = \frac{1}{\epsilon_0} \int_v \rho \, dv, \qquad (3\text{-}23)$$

- If we apply divergence theorem to the left-hand side,

$$\int_v \boldsymbol{\nabla} \cdot \mathbf{E} \, dv = \frac{1}{\epsilon_0} \int_v \rho \, dv.$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad (3\text{-}25)$$

Example-1



- A spherical charge distribytion has a Radius *R* and a uniform density ρ. Let us find *E* and *V*
- (a) The electric field strength **E**

$$Q = \tfrac{4}{3}\pi R^3 \rho. \qquad (3\text{-}26)$$

- From Gauss's Law, outside the sphere at point *P′*

$$E_o = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{R^3\rho}{3\epsilon_0 r^2} \qquad (3\text{-}27)$$

- Inside the sphere at point *P″*

$$E_i = \frac{Q(r/R)^3}{4\pi\epsilon_0 r^2} = \frac{Qr}{4\pi\epsilon_0 R^3} = \frac{\rho r}{3\epsilon_0}. \qquad (3\text{–}28)$$

- (b) The Electric potential *V* outside the sphere at point *P′*

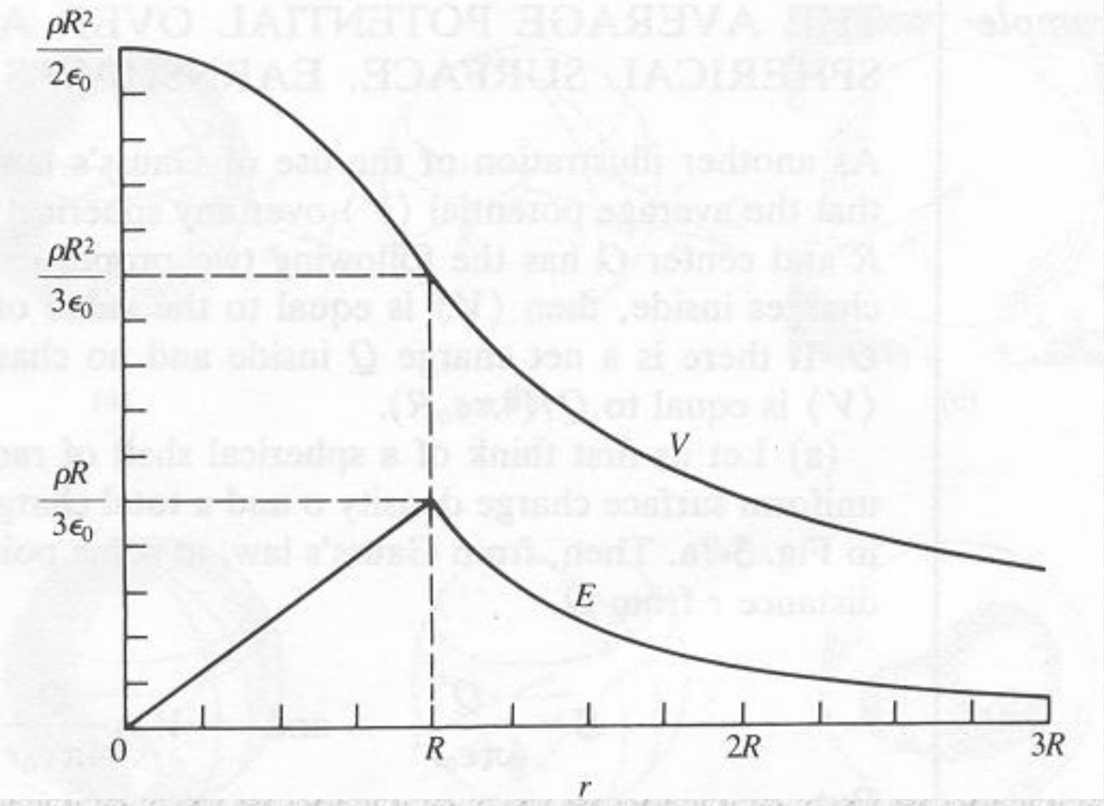
$$V_o = \frac{Q}{4\pi\epsilon_0 r} \qquad (3\text{-}29)$$

Example-1

- To find the potential, inside the sphere at point *P''*

$$V_i = \int_r^\infty E\,dr = \int_r^R E_i\,dr + \int_R^\infty E_o\,dr. \qquad (3\text{-}30)$$

$$V_i = \int_r^R \frac{Qr\,dr}{4\pi\epsilon_0 R^3} + \frac{Q}{4\pi\epsilon_0 R} = \frac{Q}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right). \qquad (3\text{-}31)$$



The Equations of Poisson and of Laplace

- Let us replace **E** by $-\nabla V$ in

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad (3\text{-}25)$$

- Then

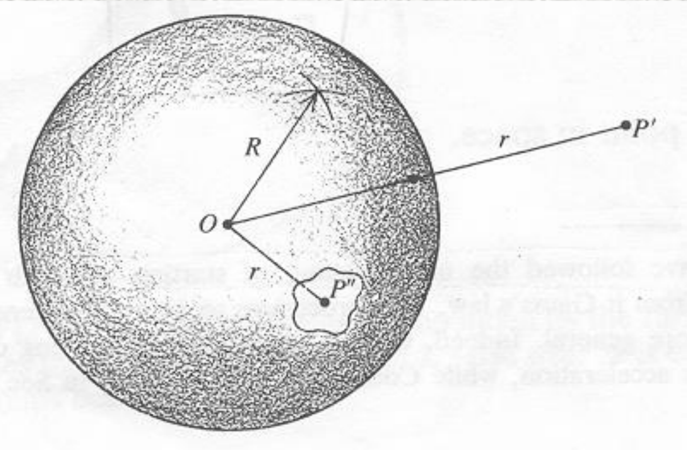
$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \qquad (4\text{-}1)$$

- This is Poisson's equation. In the region where the charge density ρ is zero,

$$\nabla^2 V = 0, \qquad (4\text{-}2)$$

- This is Laplace's equatiomn.
- To find *V* either Laplace or Poisson's equation with boundary conditions can be used.

Example-2



- A spherical charge distribytion has a Radius *R* and a uniform density ρ. Let us find *E* and *V*
- Outside the sphere, ρ=0 and

$$\nabla^2 V_o = 0. \qquad (4\text{-}3)$$

- By symmetry, *Vo* is independant of both θ and Φ, therefore,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V_o}{\partial r}\right)=0, \qquad \frac{\partial}{\partial r}\left(r^2\frac{\partial V_o}{\partial r}\right)=0, \qquad (4\text{-}4)$$

$$\frac{\partial V_o}{\partial r}=\frac{A}{r^2}, \qquad E_o=-\frac{A}{r^2}, \qquad (4\text{-}5)$$

- Inside the sphere

$$\nabla^2 V_i = -\frac{\rho}{\epsilon_0}, \qquad (4\text{-}6)$$

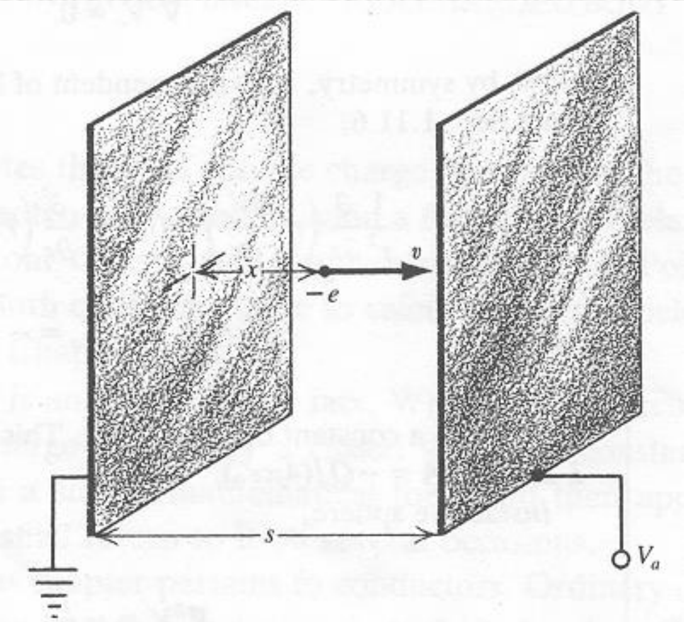
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V_i}{\partial r}\right) = -\frac{\rho}{\epsilon_0}, \qquad (4\text{-}7)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V_i}{\partial r}\right) = -\frac{\rho r^2}{\epsilon_0}, \qquad (4\text{-}8)$$

$$r^2\frac{\partial V_i}{\partial r} = -\rho\frac{r^3}{3\epsilon_0}+B, \qquad (4\text{-}9)$$

$$E_i = \frac{\rho r}{3\epsilon_0}-\frac{B}{r^2}, \qquad (4\text{-}10)$$

Example-3



- Let us consider a vacuum diode whose cathode and anode are plane, paralel and seperated by a distance *s* tha tis small compared to their linear extend.
- We assumed that the electrons have zero initial velocity and the currrent is not limited by the cathode temperature.
- Since *V* depends only on *x*, by hypothesis,Poisson equation;

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0}, \qquad (4\text{-}12)$$

- ρ is equal to $\rho = \frac{J}{\epsilon_o v}$,so

$$\frac{d^2V}{dx^2} = \frac{J}{\epsilon_0 v}, \qquad (4\text{-}13)$$

- Where *J* is the magnetute of the current density. By conversation of energy

$$\frac{mv^2}{2} = eV, \qquad (4\text{-}14)$$

- Where *m is the mass of an electron. Then*

$$\frac{d^2V}{dx^2} = \frac{J}{\epsilon_0(2eV/m)^{1/2}} \qquad (4\text{-}15)$$

$$\left(\frac{dV}{dx}\right)^2 = \frac{4J(mV/2e)^{1/2}}{\epsilon_0} + A, \qquad (4\text{-}16)$$

- *Where A* is a constant of integration. At the cathode *V* is zero and *A*= $(\frac{dV}{dx})^2$. By hypotesis *A* must be zero. Then

$$\frac{dV}{dx} = 2\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{m}{2e}\right)^{1/4} V^{1/4}, \qquad (4\text{-}17)$$

$$V^{3/4} = 1.5\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{m}{2e}\right)^{1/4} x + B. \qquad (4\text{-}18)$$

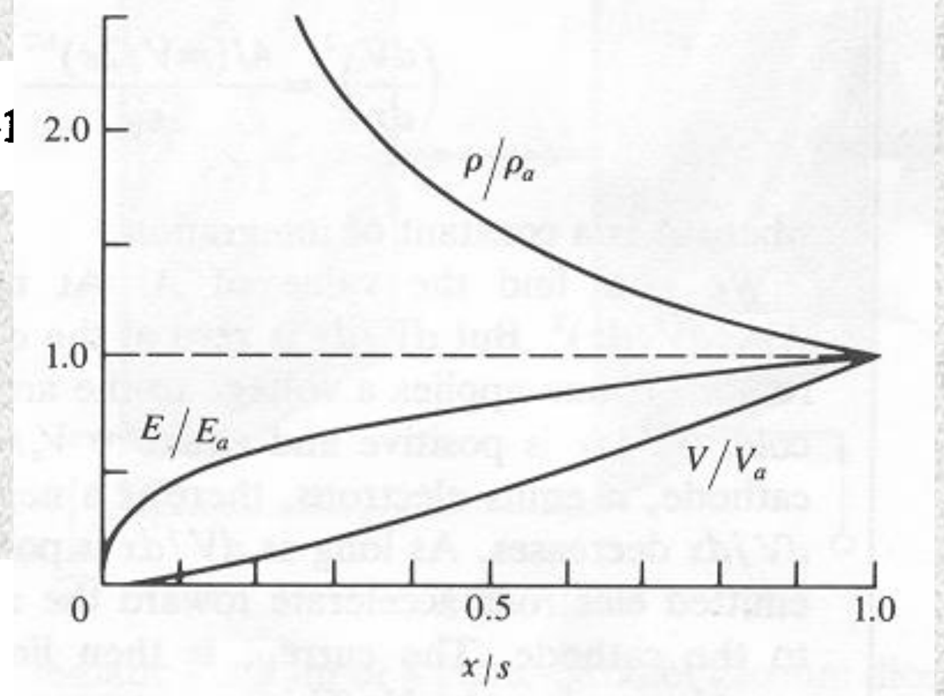
- Where *B* is the integration constant and is zero because *V* is zero at *x=0.* So

$$V = \left(\frac{9J}{4\epsilon_0}\right)^{2/3}\left(\frac{m}{2e}\right)^{1/3} s^{4/3}\left(\frac{x}{s}\right)^{4/3}. \qquad (4\text{-}1$$

- When *x=, V_a=V. Therefore*

$$V = V_a\left(\frac{x}{s}\right)^{4/3}. \qquad (4\text{-}20)$$

- Also disgarding the sign of *E,*



$$E = \frac{4}{3}\frac{V_a}{s}\left(\frac{x}{s}\right)^{1/3},$$

$$J = \frac{4\epsilon_0(2e/m)^{1/2}V_a^{3/2}}{9s^2} = 2.335 \times 10^{-6}\frac{V_a^{3/2}}{s^2} \quad \text{amperes/meter}^2, \qquad (4\text{-}22)$$

$$\rho = \frac{4\epsilon_0 V_a}{9s^2(x/s)^{2/3}}. \qquad (4\text{-}23)$$

The Law of Conversation of Electric Charge

- Consider a closed surface of area *A* enclosing a volüme *v, the volüme density is ρ.* Charges flow in and out, the current density at given point on the surface is **J**. Then

$$\int_{\mathcal{A}} \boldsymbol{J} \cdot d\boldsymbol{\mathcal{A}} = -\frac{d}{dt}\int_{v} \rho \, dv = -\frac{dQ}{dt}, \qquad (4\text{-}24)$$

- ***dA*** point aoutward. Applying divergence theorem on the left, we found

$$\int_{v} \boldsymbol{\nabla} \cdot \boldsymbol{J} \, dv = -\int_{v} \frac{\partial \rho}{\partial t} dv. \qquad (4\text{-}25)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}. \qquad (4\text{-}26)$$

- Above equations are integral and differential forms of the conservation of electric charge respectively

Conduction

- In good conductors, each atom posseses one or two conduction electrons that are free to roam about in the metarial. The current density is proportioal to the electric field.

$$\mathbf{J} = \sigma \mathbf{E}, \qquad (4\text{-}27)$$

Conductor	Conductivity σ, siemens/meter
Aluminium	3.54×10^7
Brass (65.8 Cu, 34.2 Zn)	1.59×10^7
Chromium	3.8×10^7
Copper	5.80×10^7
Gold	4.50×10^7
Graphite	7.1×10^4
Iron	1.0×10^7
Mumetal (75 Ni, 2 Cr, 5 Cu, 18 Fe)	0.16×10^7
Nickel	1.3×10^7
Seawater	~ 5
Silver	6.15×10^7
Tin	0.870×10^7
Zinc	1.86×10^7

Resistance

- If Ohm's law applies, the resistance between to electrodes fixed to a sample material is

$$R = \frac{V}{I}, \qquad (4\text{-}28)$$

- Where V is the potential difference between the two electrodes and I is the current

Conduction in a Steady Electric field

- For simplicity, we assume that the charge carries are conduction electrons. The athoms vibrate about their equilibrium positions and each electons has a kinetic energy. Then

$$\frac{mv_{th}^2}{2} = \tfrac{3}{2}kT = \tfrac{3}{2}(1.38 \times 10^{-23} \times 300) \approx 6 \times 10^{-21} \text{ joule}, \qquad (4\text{-}32)$$

$$v_{th} \approx \left(\frac{12 \times 10^{-21}}{9.1 \times 10^{-31}}\right)^{1/2} \approx 10^5 \text{ meters/second}. \qquad (4\text{-}33)$$

- Under the action of steady electric field, conduction electron drifts at a constand velocity such that

$$\mathbf{J} = \sigma \mathbf{E} = -Ne\mathbf{v}_d, \qquad (4\text{-}34)$$

- In copper $N=8.5x10^{28}$. If current of 1 amper flows through s wire having a cross section of 1 mm^2, $J=10^6$ and $v_d=10^{-4}$ m/s.

The Mobility of Conduction Electrons

- The mobility is defined as

$$\mathcal{M} = \frac{|v_d|}{E} = \frac{\sigma}{Ne} \quad (4\text{-}36)$$

- It is independent of ***E*** in linear conductors. Thus

$$\sigma = Ne\mathcal{M} \quad (4\text{-}37)$$

- The quantities *N*, *M* and σ for good conductors (gc) and semiconductors (sc) are related as follows;

$$N_{gc} \gg N_{sc}, \qquad \sigma_{gc} \gg \sigma_{sc}, \qquad \mathcal{M}_{gc} \ll \mathcal{M}_{sc}. \quad (4\text{-}39)$$

The Volume Charge Density in a Conductor-1

- (1) Under steady state conditions and a homogeneous conductor $\frac{\partial \rho}{\partial t} = 0$ then $\nabla . \boldsymbol{J} = 0$. Homogenout conductor satisfiies Ohm's law;

$$\nabla \cdot \boldsymbol{J} = \nabla \cdot \sigma \boldsymbol{E} = \sigma \nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{E} = 0. \tag{4-47}$$

- Under steady state conditions and in homogeneous conductor $\rho = 0$.
- (2) Suppose that one injects charge into a piece of copper by bombarding it with electrons

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}. \tag{4-48}$$

$$\nabla \cdot \boldsymbol{J} = \sigma \nabla \cdot \boldsymbol{E} = \frac{\sigma \rho}{\epsilon_r \epsilon_0}, \tag{4-49}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma \rho}{\epsilon_r \epsilon_0}, \qquad \rho = \rho_0 \exp\left(-\frac{\sigma t}{\epsilon_r \epsilon_0}\right), \tag{4-50}$$

The Volume Charge Density in a Conductor-2

- (3) In an homogeneous conductor carrying an alternating current, ρ is zero.
- (4) In an nonhomogeneous conductor carryig a current, ρ is not zero. Under steady-state conditions,

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\sigma \mathbf{E}) = (\nabla \sigma) \cdot \mathbf{E} + \sigma \nabla \cdot \mathbf{E} = 0 \qquad (4\text{-}51)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_r \epsilon_0} = -\frac{(\nabla \sigma) \cdot \mathbf{E}}{\sigma}. \qquad (4\text{-}52)$$

- (5) If there is a magnetic fource on the charge carries, then $J = \sigma E$ does not apply and there exists a valume charge density.

The Joule Effect

- Upon application of an electric field, the electrons gain kinetic energy between vollisions and they share tihs extra energy with the lattice. The conductor heats up. This is the **Joule effect**. The power dissipated as heat per cubic meter is;

$$P' = \frac{VI}{a^3} = \left(\frac{V}{a}\right)\left(\frac{I}{a^2}\right) = EJ \qquad (4\text{-}53)$$

$$= \sigma E^2 = \frac{J^2}{\sigma} \quad \text{watts/meter}^3. \qquad (4\text{-}54)$$

- If *E* and *J* are sinusoidal functions of the time,

$$P'_{av} = E_{rms}J_{rms} = \sigma E^2_{rms} = \frac{J^2_{rms}}{\sigma}. \qquad (4\text{-}55)$$

Isolated Conductors in Static Fields

- If one charges an isolated homogeneous conductor, the conduction electrons move about until they ahve reached their equilibrium positions and then inside the conductor, there is zero **E.**
- (1) All points inside the conductors are at same potantial
- (2) The volüme charge density is zero
- (3) Any net static charge resides on the surface of the conductor
- (4) **E** is normal at the surface of the conductor
- (5) Just outside the surface $\boldsymbol{E} = \frac{\sigma_{ch}}{\epsilon_o}$, where $\sigma_{\boldsymbol{ch}}$ is the surface charge density