EEE321 Electromagnetic Fields and Waves

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(13th Week)

Outline

- Nonuniform plane waves
- Total reflection
- Reflection and refraction at the surface of a good conductor

Nonuniform Plane Waves

- In a plane wave, the equiphase surfaces are planes.
- In a uniform plane wave, the amplitute is uniform throughout any given plane equiphase surface
- In a ununiform plane wave, the equiphase surface are still uniform but the amplitute over a given equiphase is not uniform.
- With nonuniform plane waves, we can still wirite that

$$\boldsymbol{E} = \boldsymbol{E}_m \exp j(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}), \quad (31\text{-}2)$$

$$\boldsymbol{H} = \boldsymbol{H}_m \exp j(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}), \quad (31\text{-}3)$$

- Where the amplitutes $\mathbf{E}_m$, $\mathbf{H}_m$ may be complex. The wave vector then has the form

$$\boldsymbol{k} = \boldsymbol{\beta} - j\boldsymbol{\alpha}, \quad (31\text{-}4)$$

- Where the two real vectors α and $\boldsymbol{\beta}$ points in different direction

- Then

$$\boldsymbol{E} = \boldsymbol{E}_m \exp(-\boldsymbol{\alpha} \cdot \boldsymbol{r}) \exp j(\omega t - \boldsymbol{\beta} \cdot \boldsymbol{r}), \qquad (31\text{-}5)$$

$$\boldsymbol{H} = \boldsymbol{H}_m \exp(-\boldsymbol{\alpha} \cdot \boldsymbol{r}) \exp j(\omega t - \boldsymbol{\beta} \cdot \boldsymbol{r}). \qquad (31\text{-}6)$$

- Above equations deifne a wave that propogates in the positive direction of the vector **β** at the phase velocity

$$v = \frac{\omega}{\beta} \qquad (31\text{-}7)$$

and whose wavelength is

$$\lambda = \frac{2\pi}{\beta}, \qquad \text{hence} \qquad \beta = \frac{1}{\lambda\!\!\!^{-}}. \qquad (31\text{-}8)$$

- The ampliture of the wave degreses exponantialy in the positive firection of α and the annenuation distance δ, over which the amplitute decreases by the factor of *e*, given by

$$\delta = \frac{1}{\alpha}. \qquad (31\text{-}9)$$

- Expression of **E** in the general wave equation

$$k^2 = (\boldsymbol{\beta} - j\boldsymbol{\alpha})^2 = \beta^2 - \alpha^2 - 2j\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \omega^2 \epsilon\mu - j\omega\sigma\mu, \qquad (31\text{-}10)$$

$$\beta^2 - \alpha^2 = \omega^2 \epsilon\mu, \qquad 2\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \omega\sigma\mu, \qquad (31\text{-}11)$$

Complex Angle

- From Euler formulation $\cos A + j\sin A = e^{jA}$ can be used. By extention A can be any complex number. Then

$$\sin A = \frac{\exp jA - \exp(-jA)}{2j}, \qquad \cos A = \frac{\exp jA + \exp(-jA)}{2}. \qquad (31\text{-}14)$$

- Celarly $\sin A$ can now be any complex number. The same applies to $\cos A$. As with real angles

$$\sin^2 A + \cos^2 A = 1. \qquad (31\text{-}15)$$

$$\sin(A + 2\pi) = \sin A, \qquad \cos(A + 2\pi) = \cos A. \qquad (31\text{-}16)$$

Total Reflection

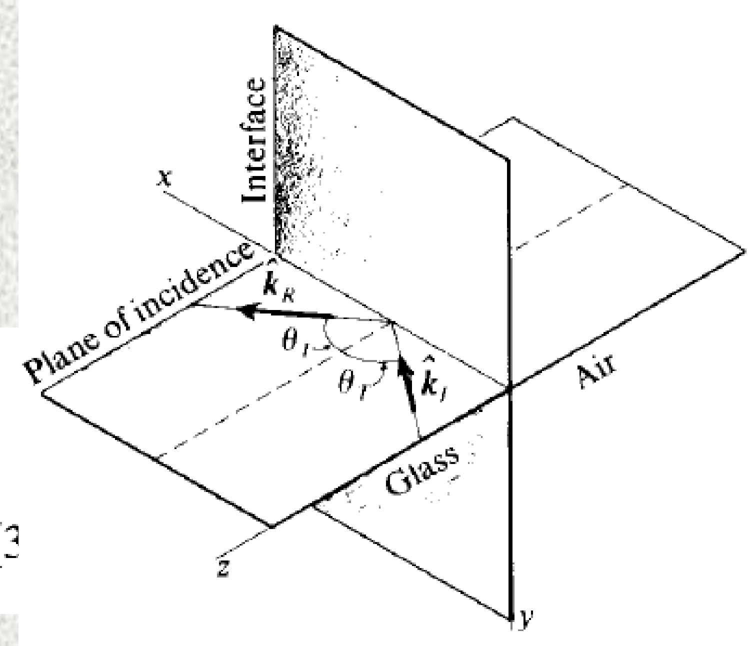
- Total reflection leads to a value of $\sin\theta_T > 1$, then θ_T is complex. Snell's law and Fressnel's equations still apply
- Critical angle of incidence, for which θ_T=90 is given by

$$\sin\theta_{Ic} = \frac{n_2}{n_1}. \qquad (31\text{-}17)$$

- At angles $\theta_I > \theta_{Ic}$, $\sin\theta_T$ is greater than unity than θ_T is complex. Then the wave is totaly reflected.
- This phenomenom is independant of the orientation of the **E** vector in the incident wave
- Total reflection has many uses (i.e optical wave guides)
- Set $\theta_T = a + jb$, then

$$\sin\theta_T = \frac{\exp j(a+jb) - \exp(-j)(a+jb)}{2j}$$

$$= \frac{\exp ja \exp(-b) - \exp(-ja)\exp b}{2j}. \qquad (3$$



- Since this quantity is real, *a* must be equal to $\pi/2$ and

$$\sin \theta_T = \frac{\exp b + \exp(-b)}{2} = \cosh b. \qquad (31\text{-}20)$$

$$\theta_T = \frac{\pi}{2} + jb. \qquad (31\text{-}21)$$

- Then

$$\cos \theta_T = \frac{\exp j(a + jb) + \exp(-j)(a + jb)}{2}$$

$$= \frac{j[\exp(-b) - \exp b]}{2} = -j \sinh b. \qquad (31\text{-}22)$$

$$\cos \theta_T = (1 - \sin^2 \theta_T)^{1/2} = -j(\sin^2 \theta_T - 1)^{1/2}. \qquad (31\text{-}23)$$

- The incident, reflected and transmitted waves are the same form asdefined in Senell's law:

$$E_I = E_{Im} \exp j[\omega t - k_1(x \sin \theta_I - z \cos \theta_I)], \qquad (31\text{-}24)$$

$$E_R = E_{Rm} \exp j[\omega t - k_1(x \sin \theta_I + z \cos \theta_I)], \qquad (31\text{-}25)$$

$$E_T = E_{Tm} \exp j[\omega t - k_2(x \sin \theta_T - z \cos \theta_T)]. \qquad (31\text{-}26)$$

The Reflected Wave

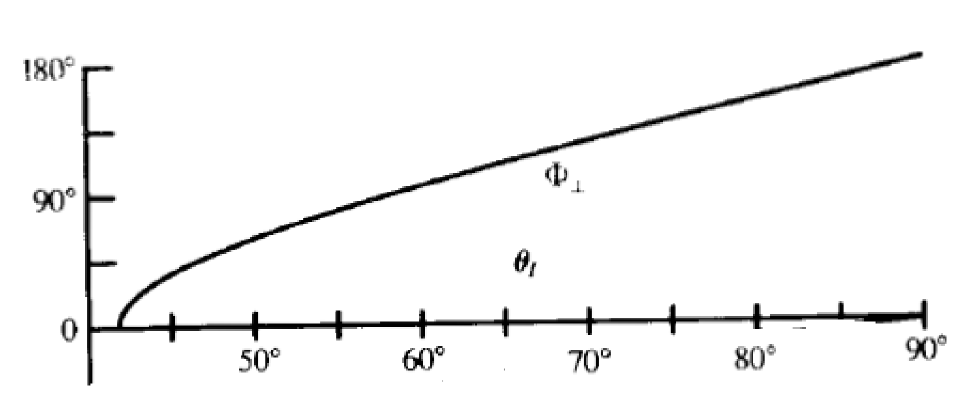
- Appliying Fresnell's equations, we find that

$$\left(\frac{E_{Rm}}{E_{Im}}\right)_{\perp} = \frac{(n_1/n_2)\cos\theta_I + j(\sin^2\theta_T - 1)^{1/2}}{(n_1/n_2)\cos\theta_I - j(\sin^2\theta_T - 1)^{1/2}} = \exp j\Phi_{\perp}, \qquad (31\text{-}27)$$

- Where

$$\Phi_{\perp} = 2\arctan\frac{(\sin^2\theta_T - 1)^{1/2}}{(n_1/n_2)\cos\theta_I}. \qquad (31\text{-}28)$$

- This is the phase of the reflected wave with respect to the incident wave at any point in the interface. The reflected wave leads the incident wavefor this polarisation
- Since total reflection is lossess, the incident and reflected waves are of the same amplitute



The Transmitted Wave

- The vector wave numver for the transmitted wave is

$$\boldsymbol{k}_T = \boldsymbol{\beta}_T - j\boldsymbol{\alpha}_T = k_2(\sin\theta_T\hat{\boldsymbol{x}} - \cos\theta_T\hat{\boldsymbol{z}}) \qquad (31\text{-}29)$$

$$= k_2(\sin\theta_T\hat{\boldsymbol{x}} + j\sinh b\hat{\boldsymbol{z}}). \qquad (31\text{-}30)$$

- This term show that the wave travels in the positive x direction and attenuation in the negative z direction because *b* is positive.
- The attenuation distance in the direction perpendicular to the interface is

$$\delta_z = \frac{1}{k_2\sinh b} = \frac{\lambda_0}{n_2\sinh b} = \frac{\lambda_0}{2\pi n_2\sinh b} = \frac{\lambda_2}{2\pi\sinh b}. \qquad (31\text{-}31)$$

- Applying Fresnell's equations

$$\left(\frac{E_{Tm}}{E_{Im}}\right)_\perp = \frac{2\cos\theta_I}{\cos\theta_I - j(n_2/n_1)(\sin^2\theta_T - 1)^{1/2}} \qquad (31\text{-}32)$$

$$= \frac{2\cos\theta_I}{[\cos^2\theta_I + (n_2/n_1)^2(\sin^2\theta_T - 1)]^{1/2}}\exp j\frac{\Phi_\perp}{2} \qquad (31\text{-}33)$$

$$= \frac{2\cos\theta_I}{[1-(n_2/n_1)^2]^{1/2}}\exp j\frac{\Phi_\perp}{2}. \qquad (31\text{-}34)$$

- Let us calculate **H**$_T$. Since the transmitted wave is not uniform, **H**$_T$ is not transverse. So

$$\boldsymbol{H}_T = \frac{\boldsymbol{k}_T \times \boldsymbol{E}_T}{\omega\mu_2} = \frac{\boldsymbol{k}_T \times \boldsymbol{E}_T}{Z_2 k_2} = \frac{\boldsymbol{k}_T \times E_T\hat{\boldsymbol{y}}}{Z_2 k_2} \qquad (31\text{-}35)$$

$$= \frac{(\sin\theta_T\,\hat{\boldsymbol{x}} - \cos\theta_T\,\hat{\boldsymbol{z}}) \times E_T\hat{\boldsymbol{y}}}{Z_2} = \frac{(\cos\theta_T\,\hat{\boldsymbol{x}} + \sin\theta_T\,\hat{\boldsymbol{z}})E_T}{Z_2}. \qquad (31\text{-}36)$$

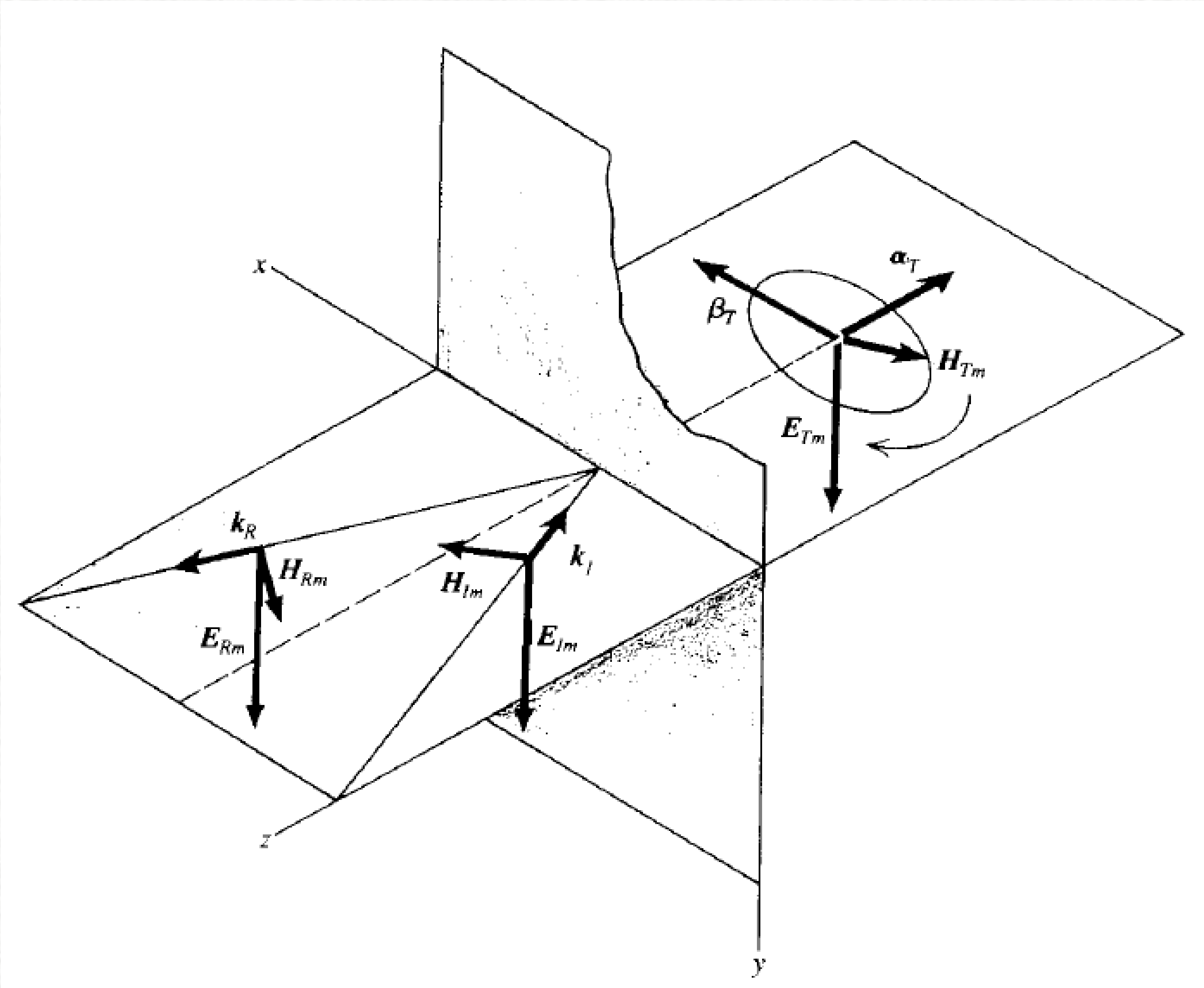
- Then $$H_{Tmx} = \frac{E_{Tm}}{Z_2}\cos\theta_T, \qquad H_{Tmz} = \frac{E_{Tm}}{Z_2}\sin\theta_T. \qquad (31\text{-}37)$$

- Since $$H_{Tm} = \frac{E_{Tm}}{Z_2}, \qquad (31\text{-}38)$$

- Then

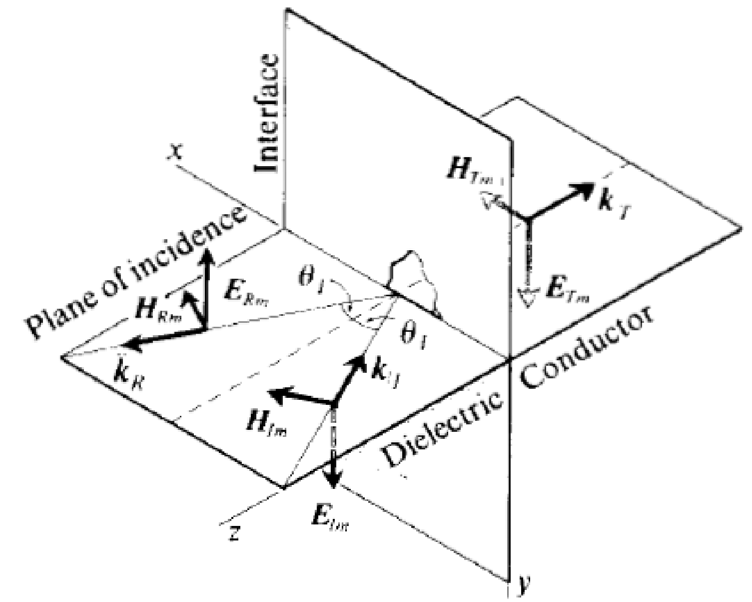
$$H_{Tmx} = H_{Tm}\cos\theta_T, \qquad H_{Tmz} = H_{Tm}\sin\theta_T, \qquad (31\text{-}39)$$

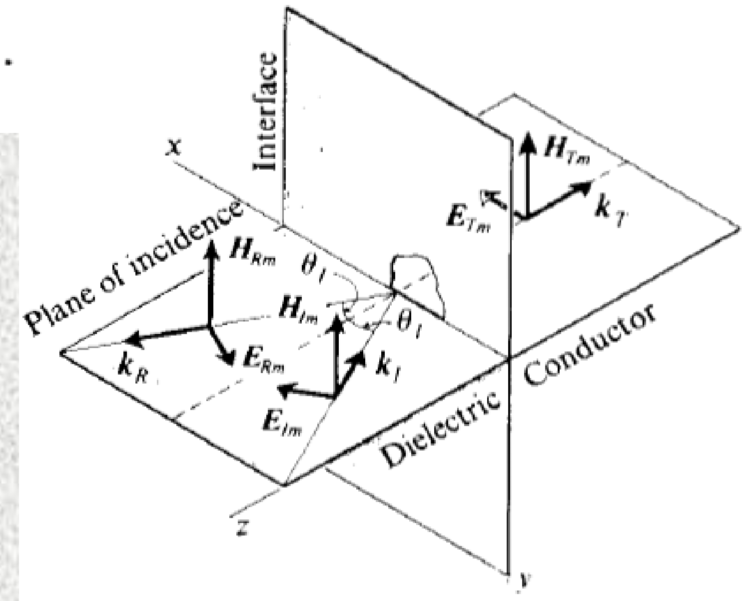
- Since $\sin\theta_T$ is real and positive while $\cos\theta_T$ is imaginary and negative the **HT** vector rotates at the angular velocity ω as in figure


x
α_T
β_T
H_{Tm}
E_{Tm}
k_R
H_{Rm}
E_{Rm}
H_{Im}
k_I
E_{Im}
z
y

Reflection and Refraction at the Surface of a Good Conductor

- Consider the second medium is a good conductor as fiven in the figures.





- We may write that

$$E_I = E_{Im} \exp j(\omega t - k_1 x \sin\theta_I + k_1 z \cos\theta_I),$$

$$E_R = E_{Rm} \exp j(\omega t - k_1 x \sin\theta_I - k_1 z \cos\theta_I),$$

$$E_T = E_{Tm} \exp j(\omega t - k_1 x \sin\theta_I + k_2 z \cos\theta_T)$$

$$= E_{Tm} \exp j\left\{\omega t - k_1 x \sin\theta_I \pm k_2 z\left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I\right]^{1/2}\right\}.$$

- From the definition

$$\frac{n_1}{n_2} = \frac{k_1}{k_2} = \frac{\omega(\epsilon_1\mu_1)^{1/2}\delta}{1-j} \qquad (32\text{-}5)$$

$$= \frac{\delta}{(1-j)\lambda\!\!\!^{-}_1}. \qquad (32\text{-}6)$$

- We shall assume that

$$\frac{\delta}{2^{1/2}\lambda_1} \ll 1 \qquad \text{or that} \qquad \left|\frac{n_1}{n_2}\right| \ll 1. \qquad (32\text{-}7)$$

- Then the expression in bracket in $\mathbf{E}_T$ equation is approximately equai to unity. Then

$$\boldsymbol{E}_T = \boldsymbol{E}_{Tm} \exp j\left(\omega t - k_1 x \sin\theta_I \pm \frac{(1-j)z}{\delta}\right) \qquad (32\text{-}8)$$

$$= \boldsymbol{E}_{Tm} \exp\left[j(\omega t - k_1 x \sin\theta_I) \pm \frac{(1+j)z}{\delta}\right]. \qquad (32\text{-}9)$$

- For reflection from a good conductor

$$\cos\theta_T = +\left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I\right]^{1/2} \approx 1, \qquad \theta_T \approx 0. \qquad (32\text{-}10)$$

$$\boldsymbol{E}_T \approx \boldsymbol{E}_{Tm} \exp\left[j\left(\omega t - k_1 x \sin\theta_I + \frac{z}{\delta}\right) + \frac{z}{\delta}\right]. \qquad \boldsymbol{E}_T \approx \boldsymbol{E}_{Tm} \exp\left[j\left(\omega t + \frac{z}{\delta}\right) + \frac{z}{\delta}\right], \qquad (32\text{-}13)$$

- The transmitted wave propogates into the conductor along the normal to the interface , whatever the angle of incidence, Also the amplitute decrease by the factor of *e* over one skin depth

E Normal to the Plane of Incidence

- From the Fresnell's equation with $\left|\frac{n_1}{n_2}\right| \ll 1$,

$$\left(\frac{E_{Rm}}{E_{Im}}\right)_\perp = \frac{(n_1/n_2)\cos\theta_I - \cos\theta_T}{(n_1/n_2)\cos\theta_I + \cos\theta_T} \approx -1 \qquad (32\text{-}14)$$

- And from the fact that $\cos\theta_T \cong 1$,

$$\left(\frac{E_{Tm}}{E_{Im}}\right)_\perp = \frac{2(n_1/n_2)\cos\theta_I}{(n_1/n_2)\cos\theta_I + \cos\theta_T} \approx 2\frac{n_1}{n_2}\cos\theta_I \approx 0, \qquad (32\text{-}15)$$

- At the surface of dielectric such that $n_2 >> n_1$ one also has that

$$\left(\frac{E_{Rm}}{E_{Im}}\right)_\perp \approx -1, \qquad \left(\frac{E_{Tm}}{E_{Im}}\right)_\perp \approx 0. \qquad (32\text{-}16)$$

E Normal to the Plane of Incidence

- From Fresnell's equation

$$\left(\frac{E_{Rm}}{E_{Im}}\right)_{\parallel} = \frac{(n_1/n_2)\cos\theta_T - \cos\theta_I}{(n_1/n_2)\cos\theta_T + \cos\theta_I}$$

$$\approx \frac{(n_1/n_2) - \cos\theta_I}{(n_1/n_2) + \cos\theta_I} \approx -1. \qquad (32\text{-}17)$$

$$\left(\frac{E_{Tm}}{E_{Im}}\right)_{\parallel} = \frac{2(n_1/n_2)\cos\theta_I}{\cos\theta_I + (n_1/n_2)\cos\theta_T} \qquad (32\text{-}18)$$

$$\approx \frac{2n_1}{n_2} = \frac{2n_1\delta}{1-j} = n_1\delta(1+j). \qquad (32\text{-}19)$$

- Above approximations are not valid at grazing incidence, where θ_I is close to 90^o.