EEE321 Electromagnetic Fields and Waves

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(11^{th} Week)

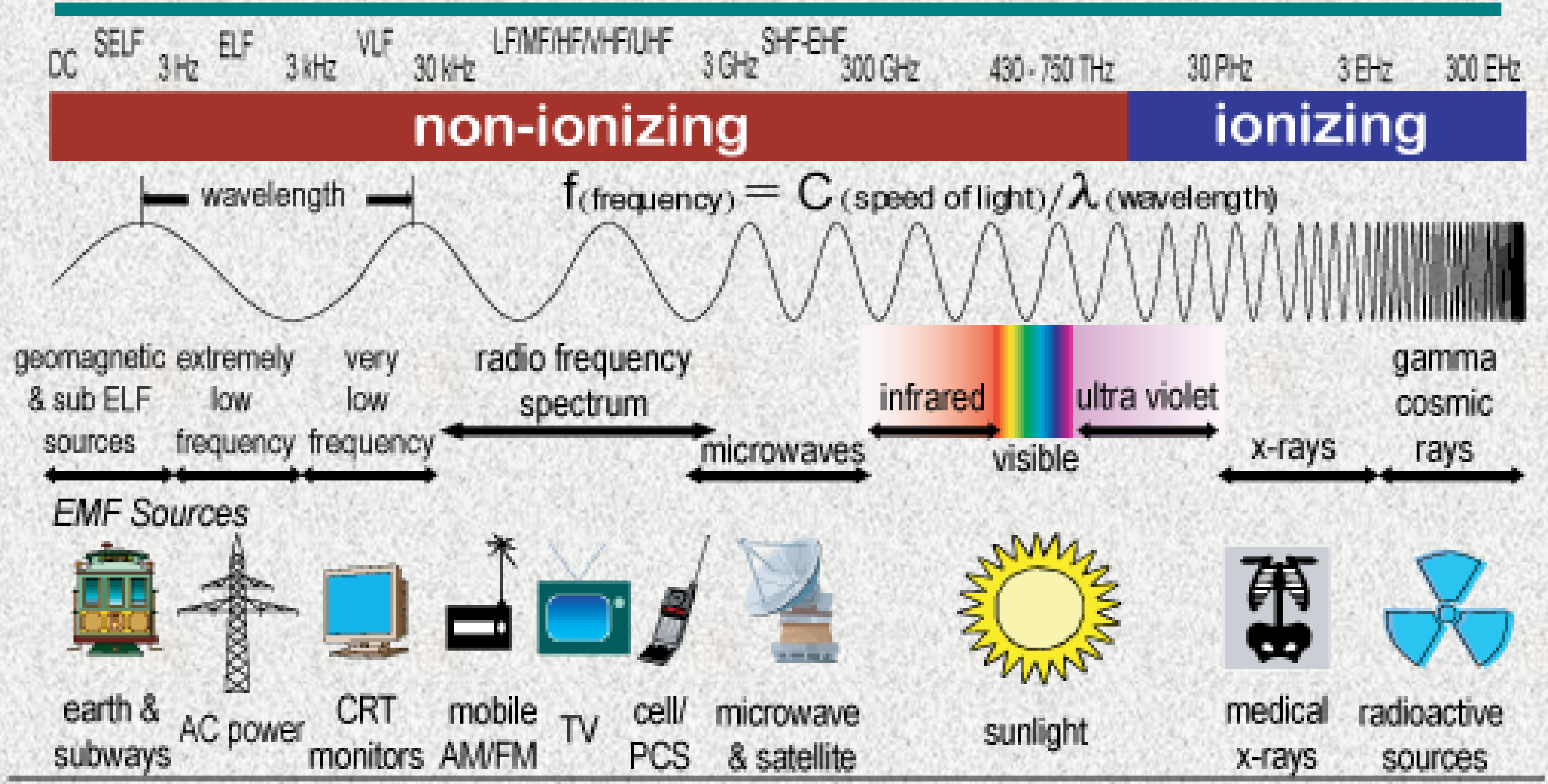
Outline

- Electromagnetic Spectrum
- Uniform Plane Electromagnetic Waves in General Medium
- Uniform Plane Waves in Free Space
- Uniform Plane Waves in Nononductor
- Uniform Plane Waves in Conductor
- Poynting Theorem

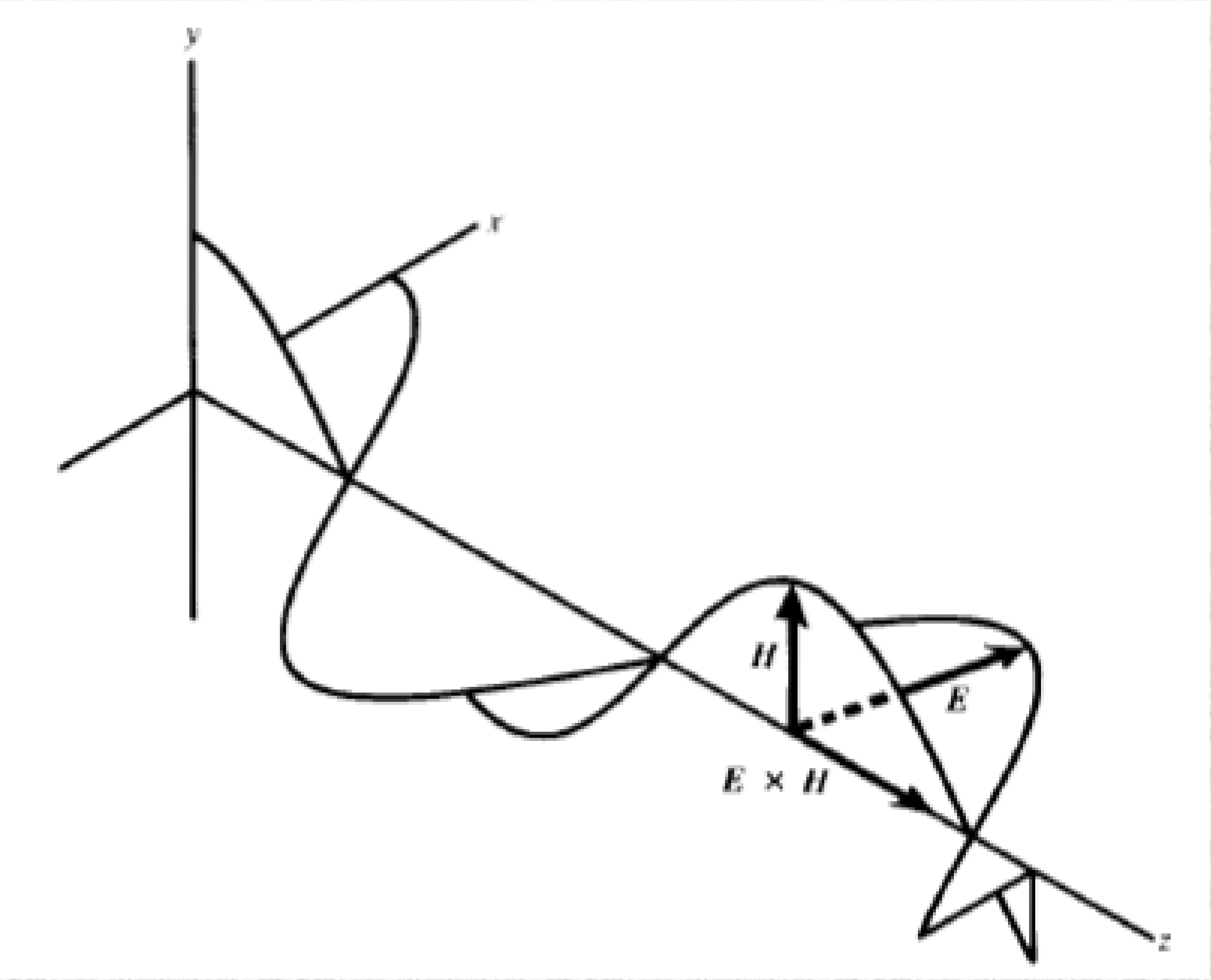
Electromagnetic Spectrum

- Maxwell’e equations impose no limit on the frequency of electromagnetic waves
- Known spectrum exteds continuously from long radio waves to the very high-energy gamma rays of cosmic radiation.
- At the beginning the frequencies were the order of 100 hertz and the wavelength about 3 megameters
- Recenly the frequencies are of the order of 10^{24} hertz and the wavelength less then 1 fentometer
- Many experiments demonstrate that all wave are transverse and they travel at the speed of light in free space
- we use **H** instead of **B** in dealing with electromagnetic waves.
  - **ExH** is power density and *E/H* is impedance

THE ELECTROMAGNETIC SPECTRUM



Gigahertz (GHz) 10-9 Terahertz (THz) 10-12 Petahertz (PHz) 10-15 Exahertz (EHz) 10-18 Zettahertz (ZHz) 10-21 Yottahertz (YHz) 10-24


y
x
H
E
E × H
z

Uniform Plane Electromagnetic Waves in a General Medium

- A wave front is a surface of uniform phase
- The wave front of a **planar wave** is planar
- A wave is uniform if a wave front is a surface of uniform phase and uniform amplitute
- In general medium that is homogeneous isotropic, linear and stationary (HILS) has parameters ϵ_r, μ_r and σ
- We assume a sinusidal wave travelling in the direction of the z-axis
- The wave is lineary polarized. So **E** and **H** are of the form

$$\boldsymbol{E} = \boldsymbol{E}_m \exp j(\omega t - kz), \qquad \boldsymbol{H} = \boldsymbol{H}_m \exp j(\omega t - kz)^{\ddagger} \qquad (28\text{-}1)$$

- Where **Em** and $\mathbf{H_m}$ are vectors that are independant of the time and of the coordinates
- If there is no attenuation, the **wave numver is real**

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda} = \frac{1}{\bar{\lambda}}, \qquad (28\text{-}2)$$

Relative Orientation of E, H, and k

- For this perticular field

$$\frac{\partial}{\partial t} = j\omega, \qquad \boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}} = \frac{\partial}{\partial z}\hat{\boldsymbol{z}} = -jk\hat{\boldsymbol{z}}. \qquad (28\text{-}3)$$

- We set ρ_f=0. We also set

$$\boldsymbol{J}_f = \sigma \boldsymbol{E}, \qquad (28\text{-}4)$$

- On the assumption that ***vxB*** is negligible compared to **E**, where ***v*** is the velocity of a conduction electron
- The Maxwel's equations then become

$$-jk\hat{\boldsymbol{z}} \cdot \boldsymbol{E} = 0, \qquad -jk\hat{\boldsymbol{z}} \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}, \qquad (28\text{-}5)$$

$$-jk\hat{\boldsymbol{z}} \cdot \boldsymbol{H} = 0, \qquad -jk\hat{\boldsymbol{z}} \times \boldsymbol{H} = \sigma\boldsymbol{E} + j\omega\epsilon\boldsymbol{E} \qquad (28\text{-}6)$$

- And then to

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{E} = 0, \qquad \boldsymbol{E} = -\frac{k}{\omega\epsilon + j\sigma}\hat{\boldsymbol{z}} \times \boldsymbol{H}, \qquad (28\text{-}7)$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{H} = 0, \qquad \boldsymbol{H} = \frac{k}{\omega\mu}\hat{\boldsymbol{z}} \times \boldsymbol{E}. \qquad (28\text{-}8)$$

Characteristic Impedance Z of a Medium

- The ratio *E/H* is the characteristic impedance Z of a medium of propogation:

$$Z = \frac{E}{H} = \frac{k}{\omega\epsilon - j\sigma} = \frac{\omega\mu}{k}. \qquad (28\text{-}9)$$

Wave Number k

- The value of k^2 follows from the equation given for impedance definition:

$$k^2 = \omega^2\epsilon\mu - j\omega\sigma\mu = \omega^2\epsilon\mu\left(1 - j\frac{\sigma}{\omega\epsilon}\right), \tag{28-10}$$

$$= \omega^2\epsilon_0\mu_0\epsilon_r\mu_r\left(1 - j\frac{\sigma}{\omega\epsilon}\right). \tag{28-11}$$

- The σ terms account for Joule losses and attenuation

Wave Equations

- We found nonhomogenous wave equation for **E** and **B** as:

$$\nabla^2 \boldsymbol{E} - \epsilon_0\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\nabla\rho}{\epsilon_0} + \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}, \qquad (28\text{-}12)$$

$$\nabla^2 \boldsymbol{B} - \epsilon_0\mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \mu_0 \nabla \times \boldsymbol{J}. \qquad (28\text{-}13)$$

- If we set $\rho_f=0$ and ϵ, μ, σ are parameters of the medium

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \mu \frac{\partial \boldsymbol{J}_f}{\partial t} = \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t}, \qquad (28\text{-}14)$$

$$\nabla^2 \boldsymbol{B} - \epsilon\mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = -\mu\sigma \nabla \times \boldsymbol{E} = \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t}. \qquad (28\text{-}15)$$

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} = 0, \qquad (28\text{-}16)$$

$$\nabla^2 \boldsymbol{B} - \epsilon\mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad (28\text{-}17)$$

- Or to use **H** instead of **B**

$$\nabla^2 \boldsymbol{H} - \epsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} = 0. \qquad (28\text{-}18)$$

- Then wave equation for **E** (**H** is also similar**)** is given by:

$$(-k^2 + \omega^2\epsilon\mu - j\omega\sigma\mu)\boldsymbol{E} = 0, \qquad (28\text{-}19)$$

Uniform Plane Electromagnetic Waves in Free Space

- In free space $\epsilon_r=1$, $\mu_r=1$, $\sigma=0$. There is no attenuation and the wave number is;

$$k = \frac{1}{\lambda_0} \tag{28-20}$$

$$= \omega(\epsilon_0\mu_0)^{1/2}. \tag{28-21}$$

- The speed of ligth is;

$$c = \frac{\omega}{k} = \frac{1}{(\epsilon_0\mu_0)^{1/2}} = 2.99792458 \times 10^8 \text{ meters/second.} \tag{28-22}$$

- The characteristic impedance of the vacuum is

$$Z_0 = \frac{E}{H} = \frac{k}{\omega\epsilon_0} = \frac{\omega\mu_0}{k} = \frac{1}{\epsilon_0 c} = \mu_0 c = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} = 3.767303 \times 10^2 \approx 377 \text{ ohms.} \tag{28-25}$$

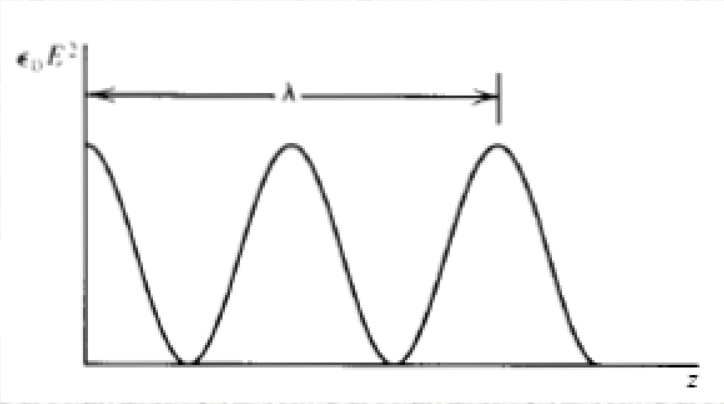
- Thus, since $B = \mu_o H$ in free space,

$$\frac{E}{B} = \frac{1}{(\epsilon_0\mu_0)^{1/2}} = c, \qquad \text{or} \qquad E = Bc. \tag{28-26}$$

- The electric and magnetic energy densities are equal:

$$\frac{\epsilon_0 E^2/2}{\mu_0 H^2/2} = \frac{\epsilon_0}{\mu_0}\left(\frac{\mu_0}{\epsilon_0}\right) = 1. \qquad (28\text{-}27)$$

- At any instand, the total energy fluctuates with z as in figure and its time averaged value at any point



$$\boldsymbol{E} = \boldsymbol{E}_m \cos(\omega t - kz), \qquad \boldsymbol{H} = \boldsymbol{H}_m \cos(\omega t - kz).$$

- Abondoning the phasor notation

$$\mathscr{E}' = \frac{\epsilon_0 E_{rms}^2}{2} + \frac{\mu_0 H_{rms}^2}{2} = \epsilon_0 E_{rms}^2 = \mu_0 H_{rms}^2. \qquad (28\text{-}28)$$

- The magnetute of the Poynting vector is

$$|\mathscr{S}| = |\boldsymbol{E} \times \boldsymbol{H}| = E_m H_m \cos^2(\omega t - kz). \qquad (28\text{-}30)$$

- Time aveared Poynting vector is

$$\mathscr{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,(\boldsymbol{E} \times \boldsymbol{H}^*) \qquad (28\text{-}31)$$

- For a uniform plane wave in free space

$$\mathcal{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,(EH^*)\hat{z} \tag{28-32}$$

$$= \tfrac{1}{2}c\epsilon_0\,|E_m|^2\,\hat{z} = c\epsilon_0 E_{rms}^2\hat{z} = \frac{E_{rms}^2}{Z_0}\,\hat{z} \tag{28-33}$$

$$\approx \frac{E_{rms}^2}{377}\,\hat{z} \qquad \text{watts/meter}^2. \tag{28-34}$$

- The time-aveaged Poyinting vector is equal to total energy density multiplied by the speed of ligth *c*

Uniform Plane Electromagnetic Waves in Nonconductor

- The stuation is the same as in free space, with ϵ and μ replacing ϵ_o and μ_o . The phase velocity is,

$$v = \frac{1}{(\epsilon\mu)^{1/2}} = \frac{c}{(\epsilon_r\mu_r)^{1/2}} = \frac{c}{n}, \quad (28\text{-}38)$$

- *n* is index of refraction:

$$n = (\epsilon_r\mu_r)^{1/2}. \quad (28\text{-}39)$$

- In nonmagnetic media

$$n = \epsilon_r^{1/2}. \quad (28\text{-}40)$$

- ϵ_r is frequency dependant so *n* is also function of frequency.
- The characteristic impedance of medium is

$$Z = \frac{E}{H} = \left(\frac{\mu}{\epsilon}\right)^{1/2} = 377\left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \text{ ohms.} \quad (28\text{-}41)$$

- The electric and magnetic energy densities are again equal

$$\frac{\epsilon E^2/2}{\mu H^2/2} = 1, \quad (28\text{-}42)$$

- Time-averaged enery density is

$$\mathscr{E}'_{av} = \frac{\epsilon E_{rms}^2}{2} + \frac{\mu H_{rms}^2}{2} = \epsilon E_{rms}^2 = \mu H_{rms}^2. \qquad (28\text{-}43)$$

- The poyinting vector **ExH** points again in the direction of

$$\mathscr{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,(EH^*)\hat{z} = \left(\frac{\epsilon}{\mu}\right)^{1/2} E_{rms}^2\hat{z} \qquad (28\text{-}44)$$

$$\mathscr{S}_{av} \approx \frac{(\epsilon_r/\mu_r)^{1/2} E_{rms}^2}{377}\hat{z} \quad \text{watts/meter}^2 \qquad (28\text{-}45)$$

$$= \frac{1}{\epsilon\mu}\,\epsilon E_{rms}^2\hat{z} = v\epsilon E_{rms}^2\hat{z}. \qquad (28\text{-}46)$$

- The time-aveaged Poyinting vector is equal to total energy density multiplied by the phase velocity

Uniform Plane Electromagnetic Waves in Conductor- Complex Wave Number ($k = \beta - j\alpha$)

- In the conductin medium

$$k^2 = \frac{\epsilon_r \mu_r}{\lambda_0^2}\left(1 - j\frac{\sigma}{\omega\epsilon}\right), \qquad (28\text{-}52)$$

- So *k* is complex. Is is the costum to set

$$k = \beta - j\alpha \quad \text{and then} \quad \mathbf{E} = \mathbf{E}_m \exp(-\alpha z) \exp j(\omega t - \beta z), \qquad (28\text{-}53)$$

- $1/\alpha$ is the attenuation distance or the skin dept δ over which the attenuation decreases by a facter of *e.* So,

$$\alpha = \frac{1}{\delta}, \qquad (28\text{-}54)$$

$$\beta = \frac{1}{\lambda} = \frac{2\pi}{\lambda}, \qquad (28\text{-}55)$$

- And phase velocity is

$$v = \frac{\omega}{\beta}. \qquad (28\text{-}56)$$

- Let us find α and β in terms of ϵ_r, μ_r and σ. First we set:

$$\mathscr{D} = \frac{\sigma}{\omega\epsilon} = \left|\frac{\sigma E}{\epsilon\, \partial E/\partial t}\right| = \left|\frac{\sigma E}{\partial D/\partial t}\right| \approx 377\,\frac{\sigma \lambda_0}{\epsilon_r}. \qquad (28\text{-}57)$$

$$\mathscr{D} = \tan l, \qquad (28\text{-}58)$$

- This is called loss angle of the medium

- Thus

$$k^2 = (\beta - j\alpha)^2 = \left(\frac{\epsilon_r \mu_r}{\lambda_0^2}\right)(1 - j\mathcal{D}), \quad (28\text{-}60)$$

$$\alpha = \frac{1}{\lambda_0}\left(\frac{\epsilon_r \mu_r}{2}\right)^{1/2}[(1 + \mathcal{D}^2)^{1/2} - 1]^{1/2}, \quad (28\text{-}61)$$

$$\beta = \frac{1}{\lambda_0}\left(\frac{\epsilon_r \mu_r}{2}\right)^{1/2}[(1 + \mathcal{D}^2)^{1/2} + 1]^{1/2}, \quad (28\text{-}62)$$

$$k = \frac{(\epsilon_r \mu_r)^{1/2}}{\lambda_0}(1 + \mathcal{D}^2)^{1/4} \exp\left(-j \arctan \frac{\alpha}{\beta}\right). \quad (28\text{-}63)$$

- For *D*<<1 (low loss dielectric)

$$\alpha \approx \frac{(\epsilon_r \mu_r)^{1/2}\mathcal{D}}{2\lambda_0} = \left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \frac{\sigma c \mu_0}{2}, \quad (28\text{-}64)$$

$$\beta \approx \frac{(\epsilon_r \mu_r)^{1/2}}{\lambda_0}, \qquad v = \frac{\omega}{\beta} \approx \frac{c}{(\epsilon_r \mu_r)^{1/2}}. \quad (28\text{-}65)$$

- For *D*>>1 (good conductor)

$$k^2 = -j\mathcal{D}\frac{\epsilon_r \mu_r}{\lambda_0^2} = -j\frac{\sigma}{\omega\epsilon}\epsilon_r \mu_r \omega^2 \epsilon_0 \mu_0 = -j\sigma\mu\omega, \quad (28\text{-}66)$$

$$k = \left(\frac{\sigma\mu\omega}{2}\right)^{1/2}(1 - j), \quad (28\text{-}67)$$

$$\alpha = \beta = \left(\frac{\sigma\mu\omega}{2}\right)^{1/2}. \quad (28\text{-}68)$$

$$n = \frac{c}{\omega/\beta} = \frac{c\beta}{\omega} = c\left(\frac{\sigma\mu}{2\omega}\right)^{1/2} \quad (28\text{-}69)$$

Uniform Plane Electromagnetic Waves in Conductor- Characteristic Impedance

- The characteristic impedance of a conducting medium is complex:

$$Z = \frac{E}{H} = \frac{k}{\omega\epsilon - j\sigma} = \frac{\omega\mu}{k} \quad (28\text{-}70)$$

$$= \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{\exp j \arctan(\alpha/\beta)}{(1+\mathscr{D}^2)^{1/4}} \approx 377\left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \frac{\exp j \arctan(\alpha/\beta)}{(1+\mathscr{D}^2)^{1/4}} \text{ ohms} \quad (28\text{-}71)$$

- This menans that *E* and *H* are not in phase:

$$\frac{E}{H} = \frac{\omega\mu}{\beta - j\alpha}, \quad (28\text{-}72) \qquad \theta = \arctan\frac{\alpha}{\beta}. \quad (28\text{-}73)$$

- Therefore

$$\boldsymbol{E} = \boldsymbol{E}_m \exp(-\alpha z) \exp j(\omega t - \beta z), \quad (28\text{-}74)$$

$$\boldsymbol{H} = \boldsymbol{H}_m \exp(-\alpha z) \exp j(\omega t - \beta z - \theta), \quad (28\text{-}75)$$

- With

$$\frac{E_m}{H_m} = \frac{\omega\mu}{k} = \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{1}{(1+\mathscr{D}^2)^{1/4}} = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \frac{1}{(1+\mathscr{D}^2)^{1/4}} \quad (28\text{-}76)$$

$$\approx 377\left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \frac{1}{(1+\mathscr{D}^2)^{1/4}} \text{ ohms.} \quad (28\text{-}77)$$

Uniform Plane Electromagnetic Waves in Conductor- Energy Density

- Time-averaged electric and magnetic energy densities are in ratio

$$\frac{\mathscr{E}'_e}{\mathscr{E}'_m} = \frac{\epsilon E^2_{\text{rms}}/2}{\mu H^2_{\text{rms}}/2} = \frac{1}{(1+\mathscr{D}^2)^{1/2}}. \qquad (28\text{-}78)$$

- There is less electric energy than magnetic energy because the conductivity both decreases **E** and adds conduction current to the displacement current, which increase **H**.
- The time-averaged total energy density is;

$$\tfrac{1}{2}(\epsilon E^2_{\text{rms}} + \mu H^2_{\text{rms}}) \exp(-2\alpha z) = \tfrac{1}{2}(\epsilon E^2_{\text{rms}})[1 + (1+\mathscr{D}^2)^{1/2}] \exp(-2\alpha z). \qquad (28\text{-}79)$$

Poyinting Theorem

- We referred to the Poyinting vector

$$\mathcal{S} = \mathbf{E} \times \mathbf{H} \qquad (28\text{-}80)$$

- This vector is of great theoretical and practical interest. To prove Poyinting Theorem, first we use vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}). \qquad (28\text{-}81)$$

- In HILS medium

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot \mu \frac{\partial \mathbf{H}}{\partial t} - \mathbf{E} \cdot \left(\epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_f \right) \qquad (28\text{-}82)$$

$$= -\frac{\partial}{\partial t} \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) - \mathbf{E} \cdot \mathbf{J}_f. \qquad (28\text{-}83)$$

- Now we change the sign and integrate over a volume *v* of a finite extent and of a surface of area *A*, and finaly apply the divergence theorem on the left then;

$$-\int_{\mathcal{A}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathcal{A} = \frac{d}{dt} \int_{v} \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) dv + \int_{v} \mathbf{E} \cdot \mathbf{J}_f \, dv. \qquad (28\text{-}84)$$

- is the total power flowing out of closed surface of area *A*

- The time-averaged magnetute of the poyinting vector is

$$\mathcal{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,\{[E_m \exp(-\alpha z)\exp j(\omega t - \beta z) \times H_m \exp(-\alpha z)\exp j(-\omega t + \beta z + \theta)]\} \qquad (28\text{-}86)$$

$$= \tfrac{1}{2} E_m H_m \cos\theta \exp(-2\alpha z), \qquad (28\text{-}87)$$

- Where θ is defined as

$$\cos\theta = \frac{\beta}{(\alpha^2 + \beta^2)^{1/2}}. \qquad (28\text{-}88)$$

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$$\mathcal{S}_{av} = \left(\frac{\epsilon}{\mu}\right)^{1/2} (1 + \mathcal{D}^2)^{1/4} E_{rms}^2 \cos\theta \exp(-2\alpha z) \qquad (28\text{-}89)$$

$$\approx \frac{1}{377}\left(\frac{\epsilon_r}{\mu_r}\right)^{1/2} (1 + \mathcal{D}^2)^{1/4} E_{rms}^2 \cos\theta \exp(-2\alpha z). \qquad (28\text{-}90)$$

$$\mathcal{S}_{av} = (\text{time-averaged energy density}) \times (\text{phase velocity}) \qquad (28\text{-}91)$$

- It is easy to show that