# EEE321 Electromagnetic Fields and Waves

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(10th Week)

# Outline

- Maxwell Equations in Differential Form
- Maxwell Equations im Integral Form
- Law of Conservation of Charge
- Duality
- Lorentz Reciprocity Theorem
- The Wave Equations for **E** and **B**

# Maxwell Equations in Differential Form

- Let us group Maxwell's equation then discuss in length later. We found them successivly in previous sections:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (27\text{-}1) \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (27\text{-}2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (27\text{-}3) \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}.^{\dagger} \quad (27\text{-}4)$$

- Above equations are general and four fundamental equation of electromagnetizm.
- They apply, whatever be the number or diversity of the soources.
- The usual coştum of writting: the field terms are on the left and the sources are on the right.
- However this is illusory because ρ and ***J*** are themselves functions of **E**and **B**.

$\boldsymbol{E}$ is the electric field strength, in volts/meter;

$\rho = \rho_f + \rho_b$ is the total electric charge density, in coulombs/meter$^3$;

$\rho_f$ is the free charge density;

$\rho_b = -\nabla \cdot \boldsymbol{P}$ is the bound charge density;

$\boldsymbol{P}$ is the electric polarization, in coulombs/meter$^2$;

$\boldsymbol{B}$ is the magnetic flux density, in teslas;

$\boldsymbol{J} = \boldsymbol{J}_f + \partial \boldsymbol{P}/\partial t + \nabla \times \boldsymbol{M}$ is the total current density, in amperes/meter$^2$;

$\boldsymbol{J}_f$ is the current density resulting from the motion of free charge;

$\partial \boldsymbol{P}/\partial t$ is the polarization current density in a dielectric;

$\nabla \times \boldsymbol{M}$ is the equivalent current density in magnetized matter;

$\boldsymbol{M}$ is the magnetization, in amperes/meter;

$c$ is the speed of light, about 300 megameters per second;

$\epsilon_0$ is the permittivity of free space, about $8.85 \times 10^{-12}$ farad/meter.

- In isotropic, linear, and stationary media;

$$\boldsymbol{J}_f = \sigma \boldsymbol{E}, \qquad \boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}, \qquad \boldsymbol{M} = \chi_m \boldsymbol{H}, \tag{27-5}$$

$$\epsilon_r = 1 + \chi_e, \qquad \mu_r = 1 + \chi_m, \tag{27-6}$$

- Where σ is conductivity, $X_e$ is the electric susceptibility, $\epsilon_r$ is the relative permittivity and $\mu_r$ is the relative permeability
- Writting out ρ and **J** in full, The Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_f - \boldsymbol{\nabla} \cdot \boldsymbol{P}}{\epsilon_0}, \tag{27-7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{27-8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{27-9}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0\left(\boldsymbol{J}_f + \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{M}\right). \tag{27-10}$$

- This is called Amperian formulation that Express the fields in terms of the four vectors **E**, **B**, **P**, and **M**.

- With homogenous, isotropic, linear and stationary (HILS) media,

$$\rho = \frac{\rho_f}{\epsilon_r} \qquad \text{(Sec. 9.9)} \tag{27-11}$$

$$\boldsymbol{P} = (\epsilon_r - 1)\epsilon_0 \boldsymbol{E} \qquad \text{(Sec. 9.9)} \tag{27-12}$$

$$\boldsymbol{M} = \frac{(\mu_r - 1)}{\mu_r \mu_0} \boldsymbol{B} \qquad \text{(Sec. 20.7)} \tag{27-13}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_f}{\epsilon}, \tag{27-14}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{27-15}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{27-16}$$

$$\nabla \times \boldsymbol{B} - \epsilon\mu \frac{\partial \boldsymbol{E}}{\partial t} = \mu \boldsymbol{J}_f. \tag{27-17}$$

- Observe that the above set of equations follows from 27.1-27.4 with following sybstitutions

$$\epsilon_0 \to \epsilon, \qquad \mu_0 \to \mu, \tag{27-18}$$

$$\rho \to \rho_f, \qquad \boldsymbol{J} \to \boldsymbol{J}_f. \tag{27-19}$$

- Sometimes it is usefull to Express Maxwell equation in terms of **E**,**D**,**B**, **H**. This formulation called Minkowski formulation of Maxwell Equations.

$$\nabla \cdot \mathbf{D} = \rho_f, \qquad (27\text{-}20) \qquad\qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad (27\text{-}21)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (27\text{-}22) \qquad\qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_j. \qquad (27\text{-}23)$$

- If the electric and magnetic fields are sinusoidal functions of time, then

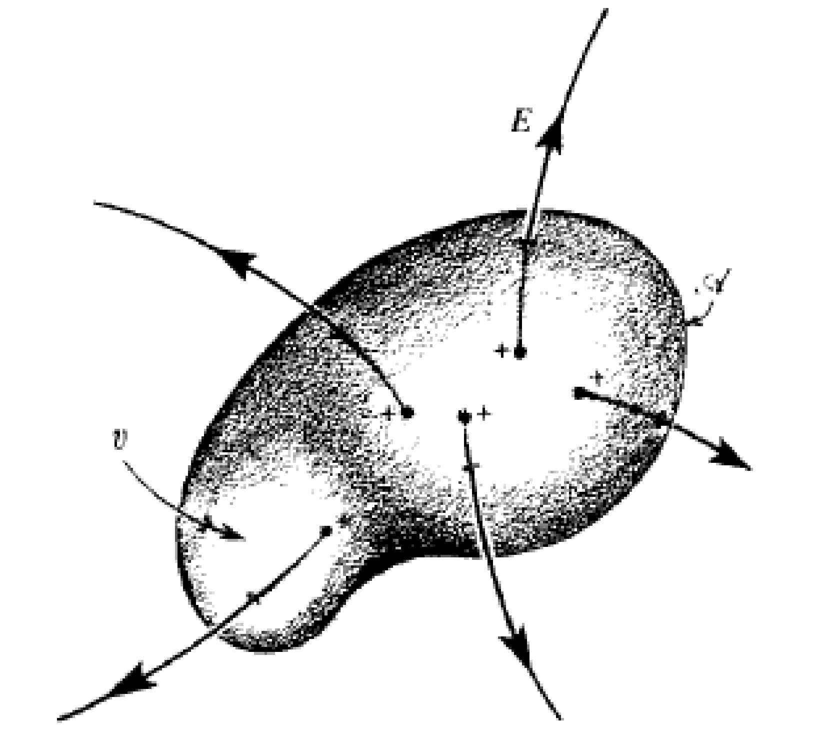
$$\nabla \cdot \epsilon \mathbf{E} = \rho_f, \qquad (27\text{-}24) \qquad\qquad \nabla \times \mathbf{E} + j\omega\mu \mathbf{H} = 0, \qquad (27\text{-}25)$$

$$\nabla \cdot \mu \mathbf{H} = 0, \qquad (27\text{-}26) \qquad\qquad \nabla \times \mathbf{H} - j\omega\epsilon \mathbf{E} = \mathbf{J}_f. \qquad (27\text{-}27)$$

# Maxwell Equations in Integral Form

- (1) Integrating $\nabla.\boldsymbol{E} = \rho/\epsilon_o$ over a finite volüme *v* and then applying the divergence theorem, we find the integral form of Gauss Law.

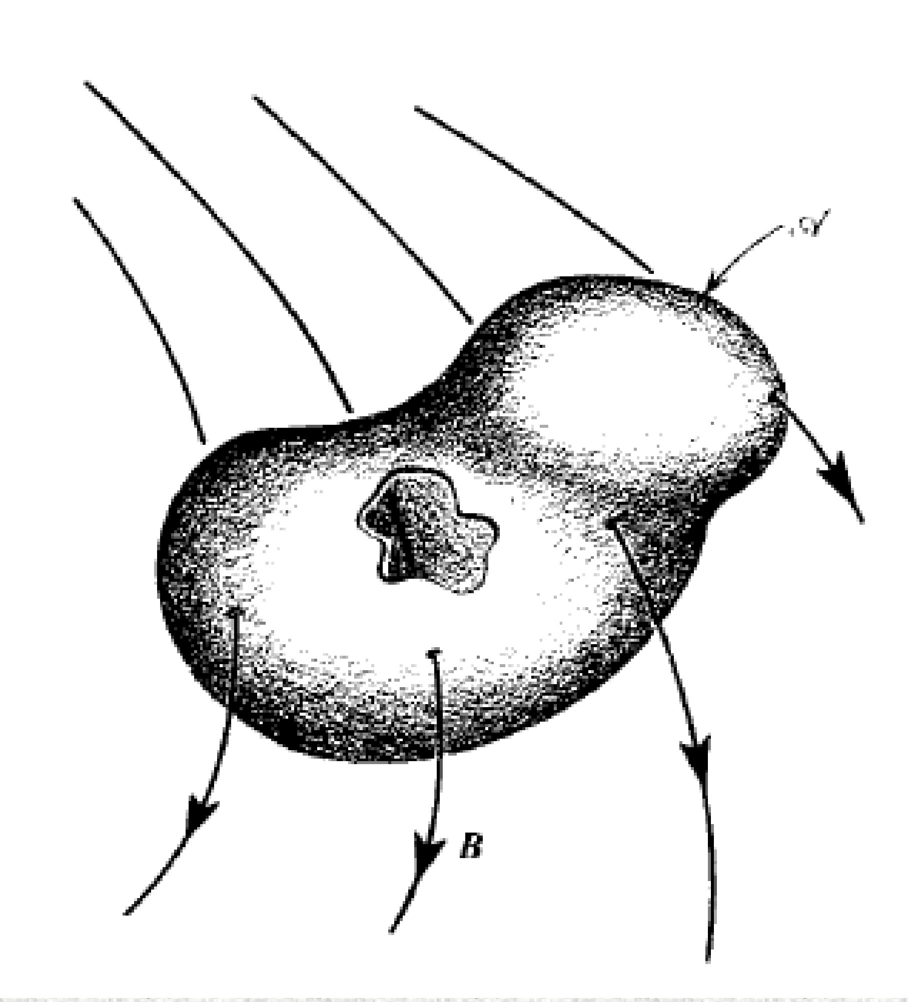
$$\int_{\mathscr{A}} \boldsymbol{E} \cdot d\boldsymbol{\mathscr{A}} = \frac{1}{\epsilon_0} \int_v \rho \, dv = \frac{Q}{\epsilon_0},$$




- Where *A* is the area of the surface bounding the volüme *v* and *Q* is the total charge enclosed within *v*.

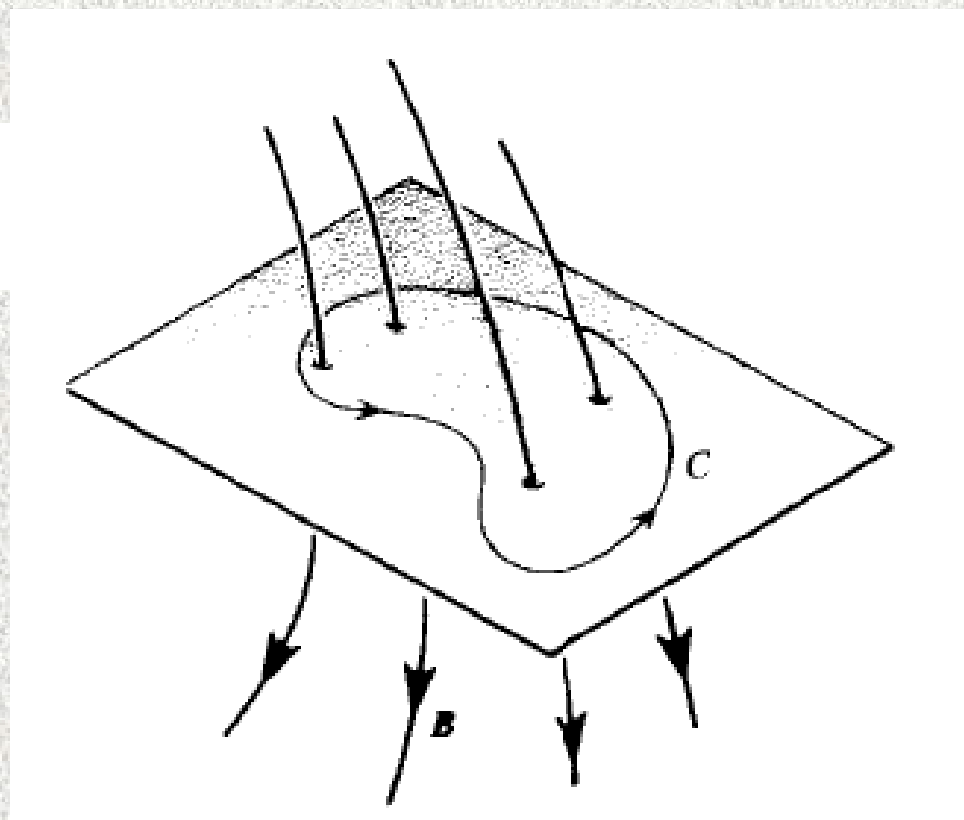
- (2) Similarly $\nabla . \boldsymbol{B} = 0$ says that the net outward flux of **B** through any closed surface is zero.

$$\int_{\mathscr{A}} \boldsymbol{B} \cdot d\mathscr{A} = 0.$$



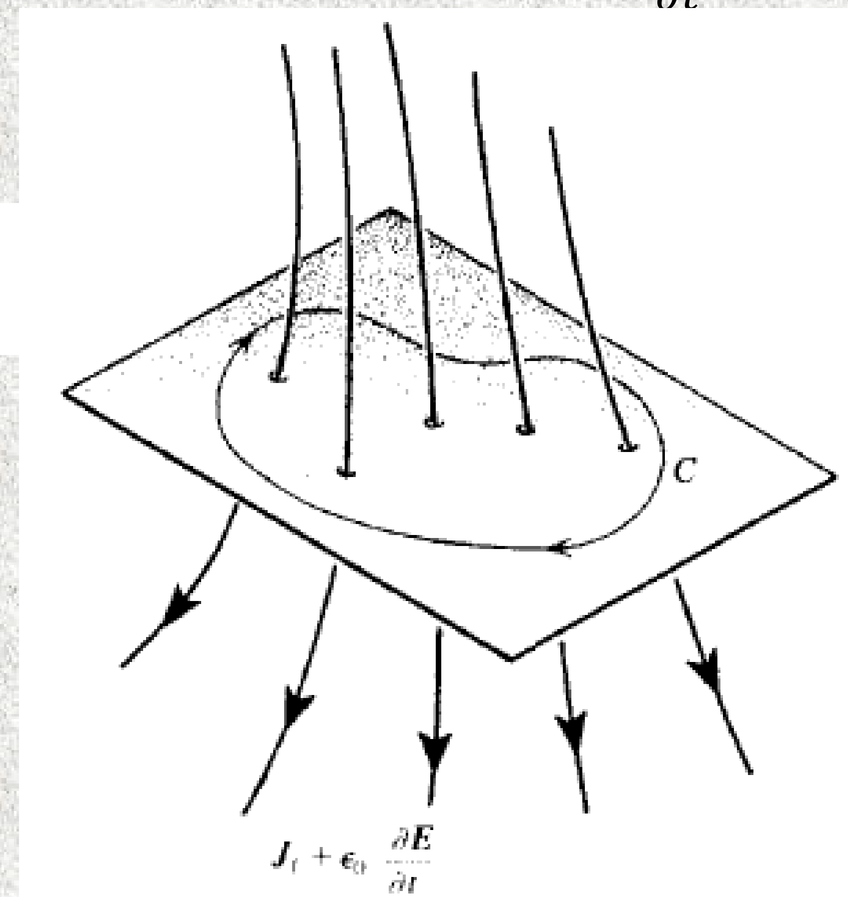
- (3) Equation $\nabla x \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} = \mathbf{0}$ is the differential form of the Faraday induction law for time-dependent magnetic fields. Integrating over an open surface of area *A* bounded by a curve *C* gives the integral form.

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{A}} \mathbf{B}\cdot d\mathcal{A} = -\frac{d\Lambda}{dt},$$



- (4) Finaly $\nabla x\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J}$ is Ampere's circuital law in integral form. The closed curve *C* bounds a surface of area *A* through which flows a current of density $\boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$.

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_{\mathscr{A}} \left( \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \cdot d\boldsymbol{\mathscr{A}}.$$



# Law of Conservation of Charge

- We saw that free charges are conserved. First

$$\nabla \cdot \mathbf{J} = \nabla \cdot \left( \mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right) = \nabla \cdot \mathbf{J}_f + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{P}), \qquad (27\text{-}41)$$

- The divergence of a curl being equal to zero. Thus

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_f}{\partial t} - \frac{\partial \rho_b}{\partial t} = -\frac{\partial (\rho_f + \rho_b)}{\partial t} = -\frac{\partial \rho}{\partial t}. \qquad (27\text{-}42)$$

- This is more general form of the law of conservation of charge.

# Redundancy in Maxwell’s Equation

- The equation for $\nabla x\boldsymbol{E}$ follows from the one for $\nabla.\boldsymbol{B}$, and the equation for $\nabla x\boldsymbol{B}$ follows from the one for $\nabla.\boldsymbol{E}$. These are, respectively the first and second pairs.
- Two equations of the first pair are related as follows;
  - If we take the divergence of $\nabla x\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} = \boldsymbol{0}$ and remember that the divergence of a curl is zero, we find that

$$\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{or} \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0. \qquad (27\text{-}43)$$

- Two equations of the second pair are related as follows;
  - Taking the divergence of $\nabla x\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J}$ and applying the law of concervation of charge we find

$$\epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = -\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t}, \qquad (27\text{-}44)$$

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right), \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} + C. \qquad (27\text{-}45)$$

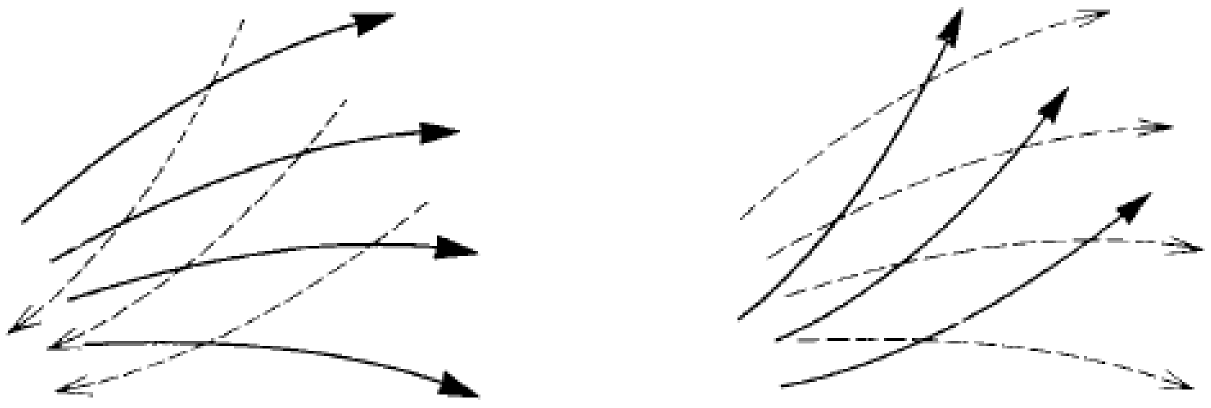
# Duality

- Imagine a field **E**, **B** that satisfy Maxwell's equations with $\rho_f=0$, $\mathbf{J}_f=0$ in a given region. The medium is HILS.
- Now imagine a different field

$$\mathbf{E}' = -K\mathbf{B} = -K\mu\mathbf{H}, \qquad (27\text{-}46)$$

$$\mathbf{H}' = +K\mathbf{D} = +K\epsilon\mathbf{E}, \qquad (27\text{-}47)$$

- Where the constant *K* has the dimension of velocity and is independent of *x*,*y*,*z*,*t*. This field also satisfy Maxwell's equations. This property is called duality of electromagnetic field.



Pair of dual fields. Lines of **E** are solid, and lines of **H** dashed.

# Lorentz Reciprocity Theorem

- Consider two fields $\boldsymbol{E}_a$, $\boldsymbol{B}_a$, and $\boldsymbol{E}_b$, $\boldsymbol{B}_b$ in a linear and isotropic medium.
- Because of the principle of superposition, these two fields can either exist seperately or being superimposed without disturbing each other, giving a third field $\boldsymbol{E}_a+\boldsymbol{E}_b$, $\boldsymbol{B}_a+\boldsymbol{B}_b$.
- If we use vector identity

$$\nabla \cdot (\boldsymbol{E}_a \times \boldsymbol{B}_b - \boldsymbol{E}_b \times \boldsymbol{B}_a) = \boldsymbol{B}_b \cdot (\nabla \times \boldsymbol{E}_a) - \boldsymbol{E}_a \cdot (\nabla \times \boldsymbol{B}_b) - \boldsymbol{B}_a \cdot (\nabla \times \boldsymbol{E}_b) + \boldsymbol{E}_b \cdot (\nabla \times \boldsymbol{B}_a). \quad (27\text{-}51)$$

- Then, from Maxwell Equations for the curls of **E** and **B**,

$$\nabla \cdot (\boldsymbol{E}_a \times \boldsymbol{B}_b - \boldsymbol{E}_b \times \boldsymbol{B}_a) = -\boldsymbol{B}_b \cdot \frac{\partial \boldsymbol{B}_a}{\partial t} - \boldsymbol{E}_a \cdot \mu_0 \left( \boldsymbol{J}_b + \epsilon_0 \frac{\partial \boldsymbol{E}_b}{\partial t} \right) + \boldsymbol{B}_a \cdot \frac{\partial \boldsymbol{B}_b}{\partial t} + \boldsymbol{E}_b \cdot \mu_0 \left( \boldsymbol{J}_a + \epsilon_0 \frac{\partial \boldsymbol{E}_a}{\partial t} \right). \quad (27\text{-}52)$$

- $$\nabla \cdot (\boldsymbol{E}_a \times \boldsymbol{B}_b - \boldsymbol{E}_b \times \boldsymbol{B}_a) = -\mu_0 \boldsymbol{E}_a \cdot \left( \boldsymbol{J}_{fb} + \frac{\partial \boldsymbol{P}_b}{\partial t} + \nabla \times \boldsymbol{M}_b \right) + \mu_0 \boldsymbol{E}_b \cdot \left( \boldsymbol{J}_{fa} + \frac{\partial \boldsymbol{P}_a}{\partial t} + \nabla \times \boldsymbol{M}_a \right). \quad (27\text{-}53)$$

- If the medium is linear, isotropic and nonmagnetic then

$$\nabla \cdot (\mathbf{E}_a \times \mathbf{B}_b - \mathbf{E}_b \times \mathbf{B}_a) = 0. \qquad (27\text{-}54)$$

- If the madium is magnetic, similar calculation can be performed by using **H** instead of **B**.

$$\nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) = 0. \qquad (27\text{-}55)$$

- Applying the divergence theorem yields

$$\int_{\mathcal{A}} (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot d\boldsymbol{\mathcal{A}} = 0, \qquad (27\text{-}56)$$

- Where *A* is the area of any closed surface, with the above ristrictions.
- This is called **Lorentz Reciprocity Theorem**.

# Wave Equations for E and for B

- Taking the curl of equation $\nabla x \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} = \mathbf{0}$ and remembering that

$$\nabla \times \nabla \times \boldsymbol{E} = -\nabla^2 \boldsymbol{E} + \nabla(\nabla \cdot \boldsymbol{E}), \qquad (27\text{-}63)$$

- Then, from equation $\nabla x \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J}$,

$$\nabla^2 \boldsymbol{E} - \nabla(\nabla \cdot \boldsymbol{E}) = \frac{\partial}{\partial t}\nabla \times \boldsymbol{B} = \frac{\partial}{\partial t}\left(\mu_0 \boldsymbol{J} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}\right). \qquad (27\text{-}64)$$

- Substituting the value of the fivergence of **E** from $\nabla . \boldsymbol{E} = \rho/\epsilon_o$ and rearranging

$$\nabla^2 \boldsymbol{E} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\nabla \rho}{\epsilon_0} + \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}. \qquad (27\text{-}65)$$

- This is nonhomogenous wave equation for **E**. The fource term is on the rigth. Outside the source

$$\nabla^2 \boldsymbol{E} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0. \qquad (27\text{-}66)$$

- Similarly, taking the curl of $\nabla x \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J}$ and substituting equation $\nabla x \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}$ and $\nabla . \boldsymbol{B} = 0$, we find

$$\nabla^2 \boldsymbol{B} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = -\mu_0 \nabla \times \boldsymbol{J}, \qquad (27\text{-}68)$$

- This is nonhomogenous wave equation for **B**. The fource term is on the rigth. Outside the source

$$\nabla^2 \boldsymbol{B} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0. \qquad (27\text{-}69)$$

- The wave equation fo HILS medium

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\nabla \rho_f}{\epsilon} + \mu \frac{\partial \boldsymbol{J}_f}{\partial t}, \qquad (27\text{-}70)$$

$$\nabla^2 \boldsymbol{B} - \epsilon\mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = -\mu \nabla \times \boldsymbol{J}_f. \qquad (27\text{-}71)$$

- If σ is constant

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = \frac{\nabla \rho_f}{\epsilon}, \qquad (27\text{-}72)$$

$$\nabla^2 \mathbf{B} - \epsilon\mu \frac{\partial^2 \mathbf{B}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{B}}{\partial t} = 0. \qquad (27\text{-}73)$$